Chebyshev Collocation Approach for a Continuous Formulation of Implicit Hybrid Methods for Vips In Second Order Odes

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Abstract: *In this paper, an implicit one-step method for numerical solution of second order Initial Value Problems of Ordinary Differential Equations has been developed by collocation and interpolation technique. The one-step method was developed using Chebyshev polynomial as basis function and, the method was augmented by the introduction of offstep points in order to bring about zero stability and upgrade the order of consistency of the new method. An advantage of the derived continuous scheme is that it can produce several outputs of solution at the off-grid points without requiring additional interpolation. Numerical examples are presented to portray the applicability and the efficiency of the method.*
Keywords: *Interpolation, Chebyshev polynomial, Collocation, continuous scheme.*

I. Introduction

The general second order Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs) of the form:

$$y'' = f(x, y, y'), y'(a) = z_o, y(a) = y_o, x \in [a,b] \tag{1}$$

where f is continuous in [a,b], is often encountered in areas such as satellite tracking/warning systems, celestial mechanics, mass action kinetics, solar systems and molecular biology [1]. Many of such problems may not be easily solved analytically, hence numerical schemes are developed to solve (1). These equations are usually reduced to a system of two first order ODEs and numerical methods for first order differential equations are used to solve it. For such systems of first order ODEs, Linear Multistep Methods (LMMs) are powerful numerical methods.

Some researchers have attempted the solution of (1) using LMMs without reduction to system of first order ODEs. They include [2], [3], [4], [5] to mention a few. [6], proposed a continuous scheme based on collocation which was found to have better error estimate and provided approximation at all interior points of the interval of consideration. The main setback of the scheme proposed by [6] is in the need to develop computer sub-programs needed to initialize the starting values; hence, much time is lost and the cost of implementation is high. In view of these disadvantages, many researchers concentrated efforts on advancing the numerical solution of IVPs in ODEs. One of the outcomes is the development of a class of methods called Block method. The method, which shall briefly be discussed in the next section simultaneously generates approximations at different grid points in the interval of integration and is less expensive in terms of the number of function evaluations compared to the LMMs or Runge-Kutta methods.

II. Block Methods

Block methods are formulated in terms of LMMs. They provide the traditional advantage of one-step methods, e.g., Runge-Kutta methods, of being self-starting and permitting easy change of step length [7]. Another important feature of the block approach is that all the discrete schemes are of uniform order and are obtained from a single continuous formula in contrast to the non-self starting predictor-corrector approach.
In what now immediately follows, we shall develop the new method with Chebyshev polynomial as basis function.

III. Development Of The Method

In this section, we intend to derive a continuous representation of a one-step method which will be used to generate the main method and other methods required to set up the block method. We set out by approximating the analytical solution of problem (1) with a Chebyshev polynomial of the form:

$$Y(x) = \sum_{j=0}^{k} a_j T_j(x) \cong y(x) \tag{2}$$

on the partition

$$a = x_0 < x_1 < \ldots < x_n < x_{n+1} < \ldots < x_N = b$$

on the integration interval [a,b], with a constant step size h, given by $h = x_{n+1} - x_n$; n = 0, 1, …, N-1.

Convectionally, we need to interpolate at at least two points to be able to approximate (1) and, to make this happen, we proceed by arbitrarily selecting an offstep point, x_{n+v}, $v\epsilon(0,1)$, in (x_n, x_{n+1}) in such a manner that the zero-stability of the main method is guaranteed. Then (2) is interpolated at x_{n+i}, i = 0, v and its second derivative is collocated at x_{n+i}, i = 0, v and 1 so as to obtain a system of five equations each of degree four i.e. k = 4 as follows:

$$\sum_{j=0}^{4} a_j T_j(x_i) = y(x) \tag{3}$$

$$\sum_{j=0}^{4} a_j T_j''(x) = f(x, y, y') \tag{4}$$

In what follows, let us arbitrarily set $v = \dfrac{1}{2}$. Then, collocating (4) at $x = x_{n+i}, i = 0, 1 \, and \, \dfrac{1}{2}$, and

interpolating (3) at $x = x_{n+i}, i = 0 \, and \, \dfrac{1}{2}$ lead to a system of equations written in the matrix form AX = B as:

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 16 & -96 & 320 \\ 0 & 0 & 16 & 0 & -64 \\ 0 & 0 & 16 & 96 & 320 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_n \\ y_{n+1} \\ h^2 f_n \\ h^2 f_{n+\frac{1}{2}} \\ h^2 f_{n+1} \end{bmatrix} \tag{5}$$

Equation (5) is solved by Gaussian elimination method to obtain the value of the unknown parameters a$_j$, j= 0 (1) 4 as follows:

$$\left. \begin{aligned} a_0 &= y_{n+\frac{1}{2}} + \frac{h^2}{256} f_n + \frac{7h^2}{128} f_{n+\frac{1}{2}} + \frac{h^2}{256} f_{n+1} \\ a_1 &= y_{n+\frac{1}{2}} - y_n + \frac{h^2}{64} f_n + \frac{5h^2}{48} f_{n+\frac{1}{2}} + \frac{h^2}{192} f_{n+1} \\ a_2 &= \frac{h^2}{192} f_n + \frac{5h^2}{96} f_{n+\frac{1}{2}} + \frac{h^2}{192} f_{n+1} \\ a_3 &= \frac{h^2}{192} f_{n+1} - \frac{h^2}{192} f_n \\ a_4 &= \frac{h^2}{768} f_n - \frac{h^2}{384} f_{n+\frac{1}{2}} + \frac{h^2}{768} f_{n+1} \end{aligned} \right\} \tag{6}$$

Substituting (6) into (2) yields a continuous implicit hybrid one-step method in the form:

$$Y(x) = \alpha_0(x) y_n + \alpha_{\frac{1}{2}}(x) y_{n+\frac{1}{2}} + h^2 \left(\sum_{j=0}^{1} \beta_j(x) f_{n+j} + \beta_{\frac{1}{2}}(x) f_{n+\frac{1}{2}} \right) \tag{7}$$

where $\alpha_j(x) \, and \, \beta_j(x)$ are continuous coefficients, $y_{n+j} = y(x_n + jh)$ is the numerical approximation of the analytical solution at x_{n+j} and $f_{n+j} = f(x_{n+j}, y_{n+j}, y'_{n+j})$.

Equation (7) yields the parameters α_j and β_j as the following continuous function of t:

$$\left. \begin{aligned} \alpha_0(t) &= -t \\ \alpha_{\frac{1}{2}}(t) &= 1 + t \\ \beta_0(t) &= \frac{h^2}{96}(3t - 2t^3 + t^4) \\ \beta_{\frac{1}{2}}(t) &= \frac{h^2}{48}(5t + 6t^2 - t^4) \\ \beta_1(t) &= \frac{h^2}{96}(-t + 2t^3 + t^4) \end{aligned} \right\} \tag{8}$$

$$where \ t = \frac{2v - h}{h} \, and \, v = x - x_n.$$

Evaluating (7) at x_{n+1}, the main method is obtained as:

$$y_{n+1} + y_n - 2y_{n+\frac{1}{2}} = \frac{h^2}{48}(f_{n+1} + 10 f_{n+\frac{1}{2}} + f_n) \tag{9}$$

To derive the block method, additional equations are needed since equation (9) alone will not be sufficient for the solution at $x_{n+\frac{1}{2}}$ and x_{n+1} to be obtained simultaneously. The additional methods can be obtained from evaluating the first derivative of equation (7):

$$Y^{'}(x) = \frac{1}{h}[\alpha_0^{'}(x)y_n + \alpha_{\frac{1}{2}}^{'}(x)y_{n+\frac{1}{2}}] + h(\sum_{j=0}^{1}\beta_j^{'}(x)f_{n+j} + \beta_{\frac{1}{2}}^{'}(x)f_{n+\frac{1}{2}}) \qquad (10)$$

at x_n, $x_{n+\frac{1}{2}}$ and x_{n+1} respectively. This yields the following discrete derivative schemes:

$$48hy_n^{'} - 96y_{n+\frac{1}{2}} + 96y_n = h^2(f_{n+1} - 6f_{n+\frac{1}{2}} - 7f_n) \qquad (11)$$

$$48hy_{n+\frac{1}{2}}^{'} - 96y_{n+\frac{1}{2}} + 96y_n = h^2(-f_{n+1} + 10f_{n+\frac{1}{2}} + 3f_n) \qquad (12)$$

$$48hy_{n+1}^{'} - 96y_{n+\frac{1}{2}} + 96y_n = h^2(9f_{n+1} + 26f_{n+\frac{1}{2}} + f_n) \qquad (13)$$

Equations (9), (11), (12) and (13) are then solved simultaneously to obtain the following explicit results:

$$\left.\begin{aligned}
y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy_n^{'} + \frac{h^2}{96}(7f_n + 6f_{n+\frac{1}{2}} - f_{n+1}) \\
y_{n+1} &= y_n + hy_n^{'} + \frac{h^2}{6}(f_n + 2f_{n+\frac{1}{2}}) \\
y_{n+\frac{1}{2}}^{'} &= y_n^{'} + \frac{h}{24}(5f_n + 8f_{n+\frac{1}{2}} - f_{n+1}) \\
y_{n+1}^{'} &= y_n^{'} + \frac{h}{6}(f_n + 4f_{n+\frac{1}{2}} + f_{n+1})
\end{aligned}\right\} \qquad (14)$$

IV.   Numerical Examples

We consider here two test problems for the efficiency and accuracy of the one-step method implemented as a block method.

1. $y'' = y'$, $y(0) = 0$, $y'(0) = -1$, $h = 0.1$

Exact Solution : $y(x) = 1 - \exp(x)$

Source :[5]

2. $y'' + 1001y' + 1000y = 0$, $y(0) = 1$, $y'(0) = -1$, $h = 0.05$

Exact Solution : $y(x) = \exp(-x)$

Source : [8]

Table 1a: Showing the exact solutions and the computed results from the proposed methods for problem 1

X	Exact Solution	The New Method(TNM)
0.1	-0.105170918	-0.105170902
0.2	-0.221402758	-0.221402723
0.3	-0.349858807	-0.34985857
0.4	-0.491824697	-0.491824433
0.5	-0.64872127	-0.648720974
0.6	-0.8221188	-0.822118466
0.7	-1.013752707	-1.013752329
0.8	-1.225540928	-1.225540498
0.9	-1.459603111	-1.45960262
1.0	-1.718281828	-1.718281267

Table 1b: Comparing the absolute errors in The New Method (TNM) to error in [5] in problem 1

X	Error in TNM, p=4, k=1	Error in [5],p=4, k=2
0.1	0.160756E-07	0.87931600E-04
0.2	0.351602E-07	0.32671800-03
0.3	0.237576E-06	0.22155640E-02
0.4	0.2646413E-06	0.48570930E-02
0.5	0.2967001E-06	0.90977340E-02

0.6	0.3343905E-06	0.14391394E-01
0.7	0.3784705E-06	0.21437918E-01
0.8	0.4304925E-06	0.29898724E-01
0.9	0.4911569E-06	0.40300719E-01
1.0	0.561459E-06	0.52552130E-01

V.   Table of Results

Table 2a: Showing the exact solutions and the computed results from the proposed methods for problem 2

X	Exact Solution	The New Method
0.1	0.90483742E+00	0.90483742+00
0.2	0.81873075E+00	0.81873075E+00
0.3	0.74081822E+00	0.74081822E+00
0.4	0.67032005E+00	0.67032005E+00
0.5	0.60653066E+00	0.60653066E+00
0.6	0.54881163E+00	0.54881164E+00
0.7	0.49658530E+00	0.49658530E+00
0.8	0.44932896E+00	0.44932896E+00
0.9	0.40656965E+00	0.40656966E+00
1.0	0.36787944E+00	0.36787944E+00

Table 2b: Comparing the absolute errors in the New Method to error in [8] in problem 2

X	Error in TNM, p=4, k=1	Error in [8],p=6, k=5
0.1	0.23596E-09	0.698677E-11
0.2	0.47798E-09	0.100275E-11
0.3	0.58172E-09	0.785878E-11
0.4	0.73564E-09	0.104778E-10
0.5	0.81263E-09	0.632212E-10
0.6	0.89403E-09	0.100508E-10
0.7	0.99141E-09	0.936336E-11
0.8	0.101722E-08	0.264766E-11
0.9	0.10406E-08	0.106793E-10
1.0	0.107144E-08	0.232731E-10

VI.   Conclusion

The desirable property of a numerical solution is to behave like the theoretical solution of the problem which can be seen in the result above. It is obvious from TABLE 1 that the new method is more efficient and accurate. However, even though the multiple finite difference method of [8] seemed to have produced a better results at most of the points of evaluation in TABLE 2b, it should be noticed that the method had step number k = 5 against the new method of step number k = 1.

Also, the investigation, through the new method reveals the viability of this approach to solve higher order problems. In view of this, we intend to extend the work to step number k = 2 and also consider more offstep points.

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