

## Numerical Solution for Two Dimensional Laplace Equation with Dirichlet Boundary Conditions

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**Abstract :** In this paper numerical technique has been used to solve two dimensional steady heat flow problem with Dirichlet boundary conditions in a rectangular domain and focuses on certain numerical methods for solving PDEs; in particular, the Finite difference method (FDM), the Finite element method (FEM) and Markov chain method (MCM) are presented by using spreadsheets. Finally the numerical solutions obtained by FDM, FEM and MCM are compared with exact solution to check the accuracy of the developed scheme

**Keywords -** Dirichlet Conditions, Finite difference Method, Finite Element Method, Laplace Equation, Markov chain Method.

### I. Introduction

Many phenomena arising in science and engineering are modeled by partial differential equations (PDEs). In such cases the quantity of interest (e.g. temperature, potential, or displacement) is a function that depends on more than one variable (typically, space variables  $x$ ,  $y$ ,  $z$  and the temporal variable  $t$ ). The heat equation, wave equation, and Laplace's equation are among the most prominent PDEs.

The process of obtaining a numerical solution to a differential equation can be viewed in the same way as conducting a lab experiment. In a lab experiment, the physical quantity, flows velocity for example, is measured at discrete points in the domain of the interest using a measurement device. A picture of the flow variation then can be constructed by connecting the measurement points allowing us to visualise the flow.

If we require the flow quantities between the measurements points, some interpolation technique can be used which may be linear or higher order interpolation. This will depend on how far the points from each other, or how accurate we require these intermediate quantities.

In the same manner, numerical techniques convert the continuous differential equation to that of finding the solution at discrete points in space which we call grid points. A full picture of the flow then can be constructed from the solution at those points. The use of spreadsheets for solving numerical analysis has been reported in the literature [1].

This paper is organized as follows. Section 2 presents formulation of two dimensional Laplace equations with dirichlet boundary conditions. Section 3 presents the finite difference method for solving Laplace equation by using spreadsheet. Section 4 presents the finite element method using spreadsheet. Section 5 presents the Markov Chain Method. Section 6 gives exact solution of Laplace equations. Section 7 compares the results obtained by each method. Finally, Section 8 gives concluding remarks.

### II. Problem Formulation

A simple case of steady state heat conduction in a rectangular domain shown in Fig.1 may be defined by two dimensional Laplace equations:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (1)$$

For  $x = [0, a]$ ,  $y = [0, b]$ , with  $a = b = 3$

Where  $u(x, y)$  is the steady State temperature distribution in the domain.

The Dirichlet boundary conditions are

$$\begin{aligned} u(0, y) &= 0, & u(a, y) &= 0 \\ u(x, 0) &= 0, & u(x, b) &= 100 \end{aligned}$$

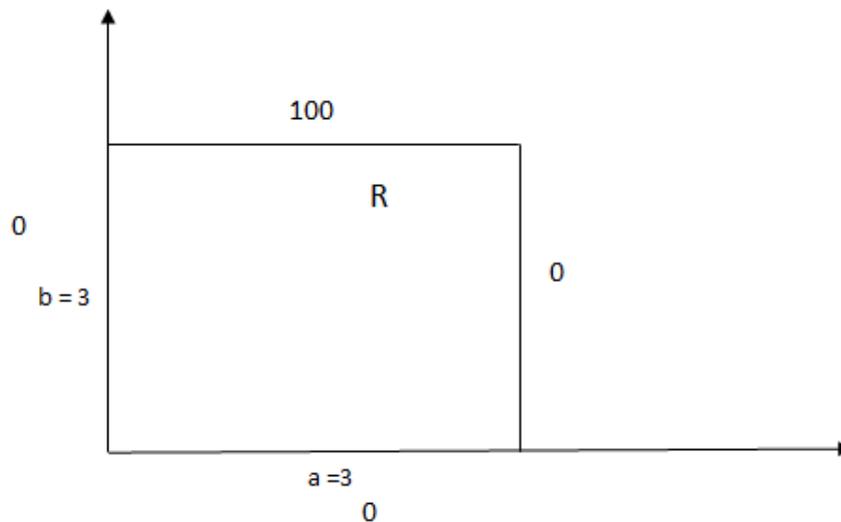


Fig. - 1: Rectangular region R with boundary conditions

The region R is divided into finite number of rectangular elements. Every node and every side of the rectangular must be common with adjacent elements except for sides on the boundaries. The nodes and elements are both numbered as shown in Fig.2

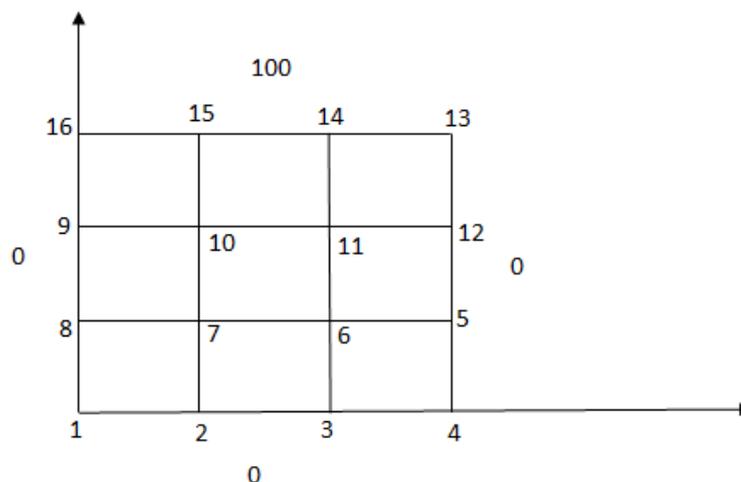


Fig. -2: Rectangular elements with nodes are numbered.

### III. The Finite Difference Method (FDM)

The finite difference method (FDM) is conceptually simple. The problems to which the method applies are specified by a PDE, a solution region (geometry), and boundary conditions. Only a brief outline of the finite difference method is given in this paper; for more detailed derivations the reader may consult [2]. The finite difference method entails three basic steps.

1. Divide the solution region into a grid of nodes. Grid points are typically arranged in a rectangular array of nodes.
2. Approximate the PDE and boundary conditions by a set of linear algebraic equations (the finite difference equations) on grid points within the solution region.
3. Solve this set of linear algebraic equations.

The region has prescribed potentials along its boundaries. The region is divided into a rectangular grid of nodes, with the numbering of free nodes as indicated in the fig. 3.

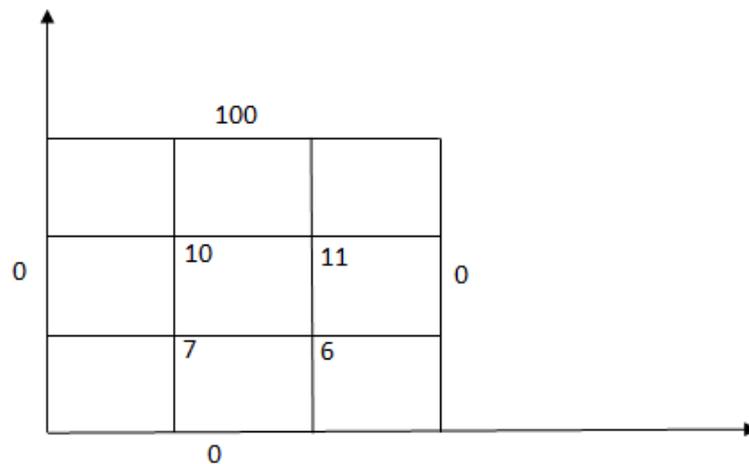


Fig.-3: The region R showing prescribed potentials at the boundaries and rectangular grid of the free nodes to illustrate the finite difference method.

Let the location of an interior grid point be identified by a pair of integers  $(i, j)$ , where  $i$  and  $j$  represent the position along the horizontal and vertical directions, respectively. For a grid having equal horizontal and vertical step sizes, the potential is given by the finite difference equation:

$$u_{i,j} = \frac{1}{4}(u_{i+1, j} + u_{i-1, j} + u_{i,j+1} + u_{i,j-1}) \tag{2}$$

The equations are formulated for all the free nodes leading to a system of linear algebraic equations. This system of equations may be solved by a variety of methods. In this paper the Gauss-Seidel method is implemented in a spreadsheet to solve this system of equations. The Gauss-Seidel method is a relatively simple iterative method for solving systems such as those encountered in the finite difference formulation. There are 4 potentials at interior grid points that need to be determined as shown in Table 1.

Table 1: The Gauss-Seidel iterations to illustrate the finite difference method.

Iteration	6	7	10	11
1	0.0	0.0	0.0	0.0
2	0.0	0.0	25.0	25.0
3	6.3	6.3	31.3	31.3
4	9.4	9.4	34.4	34.4
5	10.9	10.9	35.9	35.9
6	11.7	11.7	36.7	36.7
7	12.1	12.1	37.1	37.1
8	12.3	12.3	37.3	37.3
9	12.4	12.4	37.4	37.4
10	12.5	12.5	37.5	37.5

#### IV. The Finite Element Method (FEM)

The finite element method (FEM) is a numerical technique for solving PDEs. FEM was originally applied to problems in structural mechanics. The finite element analysis involves four basic steps.

1. Divide the solution region into a finite number of elements. The most common elements have triangular or quadrilateral shapes. The collection of all elements should resemble the original region as closely as possible.
2. Derive governing equations for a typical element. This step will determine the element coefficient matrix.
3. Assemble all elements in the solution region to obtain the global coefficient matrix.
4. Solve the resulting system of equations.

The region is divided into 18 equal triangular elements as indicated in Fig. 4. The elements are identified by encircled numbers 1 through 18. In this discretization there are 16 global nodes numbered 1 through 16 as indicated in the fig. 4. Only a brief outline of FEM is provided; for detailed derivations the reader may consult [2].

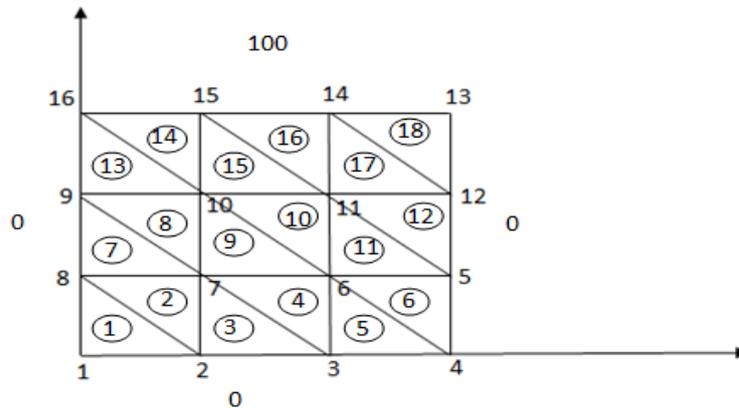


Fig.-4: The region R showing prescribed potentials at the boundaries and triangular grid of the free nodes to illustrate the finite element method.

The creation of the assembled equations required element coefficient matrix and global coefficient matrix. The entries of the  $3 \times 3$  element coefficient matrix are then given by the equation:

$$C_{ij}^{(e)} = \frac{1}{4A} [P_i P_j + Q_i Q_j] \quad (3)$$

Where  $i, j = 1, 2, 3$

$P_i, P_j$  and  $Q_i, Q_j$  for element  $e$  that are computed.

$$A = \frac{1}{2} [P_2 Q_3 - P_3 Q_2]$$

The element coefficient matrices computed according to equation (3) are, respectively as shown in Table 2

Table 2: Computation of element coefficient matrices with respective element

Element	Element Coefficient Matrix			Element	Element Coefficient Matrix		
1	1	-0.5	-0.5	10	0.5	-0.5	0
	-0.5	0.5	0		-0.5	1	-0.5
	-0.5	0	0.5		0	-0.5	0.5
2	0.5	-0.5	0	11	1	-0.5	-0.5
	-0.5	1	-0.5		-0.5	0.5	0
	0	-0.5	0.5		-0.5	0	0.5
3	1	-0.5	-0.5	12	0.5	-0.5	0
	-0.5	0.5	0		-0.5	1	-0.5
	-0.5	0	0.5		0	-0.5	0.5
4	0.5	-0.5	0	13	1	-0.5	-0.5
	-0.5	1	-0.5		-0.5	0.5	0
	0	-0.5	0.5		-0.5	0	0.5
5	1	-0.5	-0.5	14	0.5	-0.5	0
	-0.5	0.5	0		-0.5	1	-0.5
	-0.5	0	0.5		0	-0.5	0.5

6	0.5	-0.5	0	15	1	-0.5	-0.5
	-0.5	1	-0.5		-0.5	0.5	0
	0	-0.5	0.5		-0.5	0	0.5
7	1	-0.5	-0.5	16	0.5	-0.5	0
	-0.5	0.5	0		-0.5	1	-0.5
	-0.5	0	0.5		0	-0.5	0.5
8	0.5	-0.5	0	17	1	-0.5	-0.5
	-0.5	1	-0.5		-0.5	0.5	0
	0	-0.5	0.5		-0.5	0	0.5
9	1	-0.5	-0.5	18	0.5	-0.5	0
	-0.5	0.5	0		-0.5	1	-0.5
	-0.5	0	0.5		0	-0.5	0.5

The global coefficient matrix is then assembled from the element coefficient matrices. Since there are 16 nodes, the global coefficient matrix will be a **16 × 16** matrix. The assembly of the global coefficient matrix is shown in Matrix Table 3.

Table 3: Showing global coefficient matrix C

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	-0.5	0	0	0	0	0	-0.5	0	0	0	0	0	0	0	0
2	-0.5	2	-0.5	0	0	0	-1	0	0	0	0	0	0	0	0	0
3	0	-0.5	2	-0.5	0	-1	0	0	0	0	0	0	0	0	0	0
4	0	0	-0.5	1	-0.5	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	-0.5	2	-1	0	0	0	0	0	-0.5	0	0	0	0
6	0	0	-1	0	-1	4	-1	0	0	0	-1	0	0	0	0	0
7	0	-1	0	0	0	-1	4	-1	0	-1	0	0	0	0	0	0
8	-0.5	0	0	0	0	0	-1	2	-0.5	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	-0.5	2	-1	0	0	0	0	0	-0.5
10	0	0	0	0	0	0	-1	0	-1	4	-1	0	0	0	-1	0
11	0	0	0	0	0	-1	0	0	0	-1	4	-1	0	-1	0	0
12	0	0	0	0	-0.5	0	0	0	0	0	-1	2	-0.5	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	-0.5	1	-0.5	0	0
14	0	0	0	0	0	0	0	0	0	0	-1	0	-0.5	2	-0.5	0
15	0	0	0	0	0	0	0	0	0	-1	0	0	0	-0.5	2	-0.5
16	0	0	0	0	0	0	0	0	-0.5	0	0	0	0	0	-0.5	1

Defining the vector of potentials  $u_f$  and  $u_p$ , where the subscripts f and p refer to nodes with free (unknown) potentials and prescribed potentials respectively, the global coefficient matrix is then partitioned accordingly and unknown potentials are obtained from

$$u_f = -C_{ff}^{-1}C_{fp}u_p \quad (4)$$

The essential boundary conditions on the boundary of the domain, the vector of prescribed potentials  $u_p$  and the Matrices  $C_{ff}$  (Free Nodes Matrix),  $C_{fp}$  (Free and Prescribed Nodes Matrix) obtained from global coefficient matrix as shown in Table 4.

Table 4: Nodes in the finite element mesh having prescribed potentials

Node	Prescribed Potentials	Node	Prescribed Potentials
1	0	9	0
2	0	12	0
3	0	13	50
4	0	14	100
5	0	15	100
8	0	16	50

Table 5: Matrices  $C_{ff}$  and  $C_{fp}$  obtained from global coefficient matrix C.

Node	6	7	10	11
6	4	-1	0	-1
7	-1	4	-1	0
10	0	-1	4	-1
11	-1	0	-1	4

Matrix of free nodes  $C_{ff}$

Node	1	2	3	4	5	8	9	12	13	14	15	16
6	0	0	-1	0	-1	0	0	0	0	0	0	0
7	0	-1	0	0	0	-1	0	0	0	0	0	0
10	0	0	0	0	0	0	-1	0	0	0	-1	0
11	0	0	0	0	0	0	0	-1	0	-1	0	0

Matrix of free and prescribed Nodes  $C_{fp}$

The implementation of Equation (4) has been broken down into three parts.

1. Computation of the inverse of the  $C_{ff}$  matrix (this has been labelled  $A^{-1} = C_{ff}^{-1}$ )
2. Computation of an intermediate vector  $b = -C_{fp}u_p$  and
3. Computation of vector of potentials at free nodes  $u_f = A^{-1}b$

This step is shown in below Table 6.

Node	Inverse				Node	Vector	Node	Vector
6	0.2917	0.0833	0.0417	0.0833	6	0	6	12.5
7	0.0833	0.2917	0.0833	0.0417	7	0	7	12.5
10	0.0417	0.0833	0.2917	0.0833	10	100	10	37.5
11	0.0833	0.0417	0.0833	0.2917	11	100	11	37.5

Table 6: Final Calculation: (left) the inverse of the  $C_{ff}$  matrix, (middle) the intermediate vector  $b$ , (right) the vector of potentials at free nodes  $u_f$

### V. Markov Chains Method (MCM)

A Markov process is a type of random process that is characterized by the memory less property. It is a process evolving in time that remembers only the most recent past and whose conditional probability distributions are time invariant. Markov Chains are mathematical models of this kind of process. The Markov Chain is the random walk and the states are the grid nodes.

In this method, if we assume that there are  $f$  free nodes (non-absorbing) and  $p$  fixed nodes (absorbing), the size of the transition matrix  $P$  is  $n$ . Where  $n = f + p$ . Only a brief outline of the Markov Chain Method is given in this paper; for more detailed the reader may consult [3]. The Markov chains analysis involves four basic steps.

1. If the absorbing nodes and the non-absorbing nodes are numbered, then the  $n \times n$  transition matrix becomes

$$P = \begin{bmatrix} I & 0 \\ R & Q \end{bmatrix} \quad (5)$$

Where the  $f \times p$  matrix  $R$  represents the probabilities of moving from non-absorbing nodes to absorbing ones, the  $f \times f$  matrix  $Q$  represents the probabilities of moving from one non absorbing node to another, the  $p \times p$  identity matrix  $I$  represents transitions between the absorbing nodes and 0 is the null matrix showing that there are no transitions from absorbing to non-absorbing nodes.

2. The probability matrix  $B$  is  $B = [R \ Q]$
3. If  $u_f$  and  $u_n$  contain potentials at the free and fixed nodes respectively, then  $u_f = Bu_n$ , where  $f$  is the number of free nodes.
4. Solving these equations by iterative method we get the solution of free nodes  $u_f$ .

The region has prescribed potentials along its boundaries. The region is divided into a rectangular grid of nodes, with the numbering of free nodes (non-absorbing) and prescribed nodes (absorbing) as indicated in the fig.5.

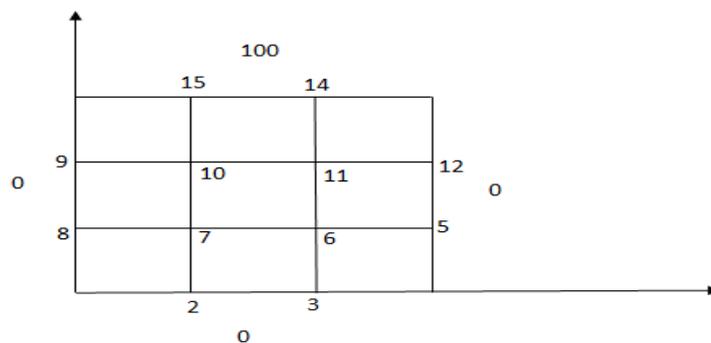


Fig.-5: The region R showing prescribed potentials at the boundaries and rectangular grid of the free nodes to illustrate the Markov Chain method.

To apply Markov Chain Technique, we number the nodes as in figure. There are four free nodes ( $f = 4$ ) and eight fixed nodes ( $p = 8$ ). The transition probability matrix  $P$  is shown in Matrix table 7.

Table 7: Showing the transition probability matrix P

	2	3	5	6	7	8	9	10	11	12	14	15
2	1	0	0	0	0	0	0	0	0	0	0	0
3	0	1	0	0	0	0	0	0	0	0	0	0
5	0	0	1	0	0	0	0	0	0	0	0	0
6	0	0.25	0.25	0	0.25	0	0	0	0.25	0	0	0
7	0.25	0	0	0.25	0	0.25	0	0.25	0	0	0	0
8	0	0	0	0	0	1	0	0	0	0	0	0
9	0	0	0	0	0	0	1	0	0	0	0	0
10	0	0	0	0.25	0.25	0	0.25	0	0.25	0	0	0.25
11	0	0	0	0.25	0	0	0	0.25	0	0.25	0.25	0
12	0	0	0	0	0	0	0	0	0	1	0	0
14	0	0	0	0	0	0	0	0	0	0	1	0
15	0	0	0	0	0	0	0	0	0	0	0	1

Defining the vector of potentials  $u_f$  and  $u_n$ , where the subscripts  $f$  and  $n$  refer to nodes with free (non-absorbing or unknown) potentials and prescribed (absorbing) potentials respectively, the transition probability matrix is then partitioned accordingly and unknown potentials are obtained from

$$u_f = Bu_n \quad (6)$$

Where  $B = [R \ Q]$

The essential boundary conditions on the boundary of the domain, the Matrices Q (non absorbing Nodes Matrix), R (non-absorbing and absorbing Nodes Matrix) obtained from the transition probability matrix as shown in table 8 and Probability Matrix B as shown in Matrix table 9.

Table 8: Matrices Q and R obtained from the transition probability matrix P

Node	6	7	10	11
6	0	0.25	0	0.25
7	0.25	0	0.25	0
10	0	0.25	0	0.25
11	0.25	0	0.25	0

Matrix of the non-absorbing nodes Q

Node	2	3	5	8	9	12	14	15
6	0	0.25	0.25	0	0	0	0	0
7	0.25	0	0	0.25	0	0	0	0
10	0	0	0	0	0.25	0	0	0.25
11	0	0	0	0	0	0.25	0.25	0

Matrix of the non-absorbing and absorbing nodes R

Table 9: Showing the Probability Matrix B

Nodes	Non-absorbing Nodes								Absorbing Nodes			
	2	3	5	8	9	12	14	15	6	7	10	11
6	0	0.25	0.25	0	0	0	0	0	0	0.25	0	0.25
7	0.25	0	0	0.25	0	0	0	0	0.25	0	0.25	0
10	0	0	0	0	0.25	0	0	0.25	0	0.25	0	0.25
11	0	0	0	0	0	0.25	0.25	0	0.25	0	0.25	0

The implementation of Equation (6) has been broken down into four sub-equations:

$$\begin{aligned}
 u_6 &= \frac{1}{4} [u_{11} + u_7 + 0 + 0] \\
 u_7 &= \frac{1}{4} [u_{10} + 0 + 0 + u_6] \\
 u_{10} &= \frac{1}{4} [100 + 0 + u_7 + u_{11}] \\
 u_{11} &= \frac{1}{4} [100 + u_{10} + u_7 + 0]
 \end{aligned}$$

Solving the above equations by the Gauss-Seidel iterations method, we get

Table 10 : The Gauss-Seidel iterations to illustrate the Markov Chain Method.

Iteration	6	7	10	11
1	0.0	0.0	0.0	0.0
2	0.0	0.0	25.0	25.0
3	6.3	6.3	31.3	31.3
4	9.4	9.4	34.4	34.4
5	10.9	10.9	35.9	35.9
6	11.7	11.7	36.7	36.7
7	12.1	12.1	37.1	37.1
8	12.3	12.3	37.3	37.3
9	12.4	12.4	37.4	37.4
10	12.5	12.5	37.5	37.5

**VI. Exact Solution**

The Exact solution of two dimensional Laplace equations with Dirichlet boundary conditions obtained by the separation of variables is given by the equation:

$$u(x,y) = \frac{400}{\pi} \sum_{k=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)}{n \sinh(n\pi)}, n = 2k - 1 \tag{7}$$

**VII. Comparison between Numerical solution by FDM, FEM and MCM and Exact Solution**

As indicated in Table 11, the potentials at the free nodes computed by FDM, FEM and MCM numerical solutions compared fairly well. The better agreement should be obtained between the all numerical solution results by using a rectangular grid for finite difference and Markov chain solution and by using triangular grid for finite element solution.

Table 11: Comparison between Numerical and Exact Solution

Node	Numerical Solution by FDM,FEM and MCM	Exact solution
6	12.5	11.926
7	12.5	11.926
10	37.5	38.074
11	37.5	38.074

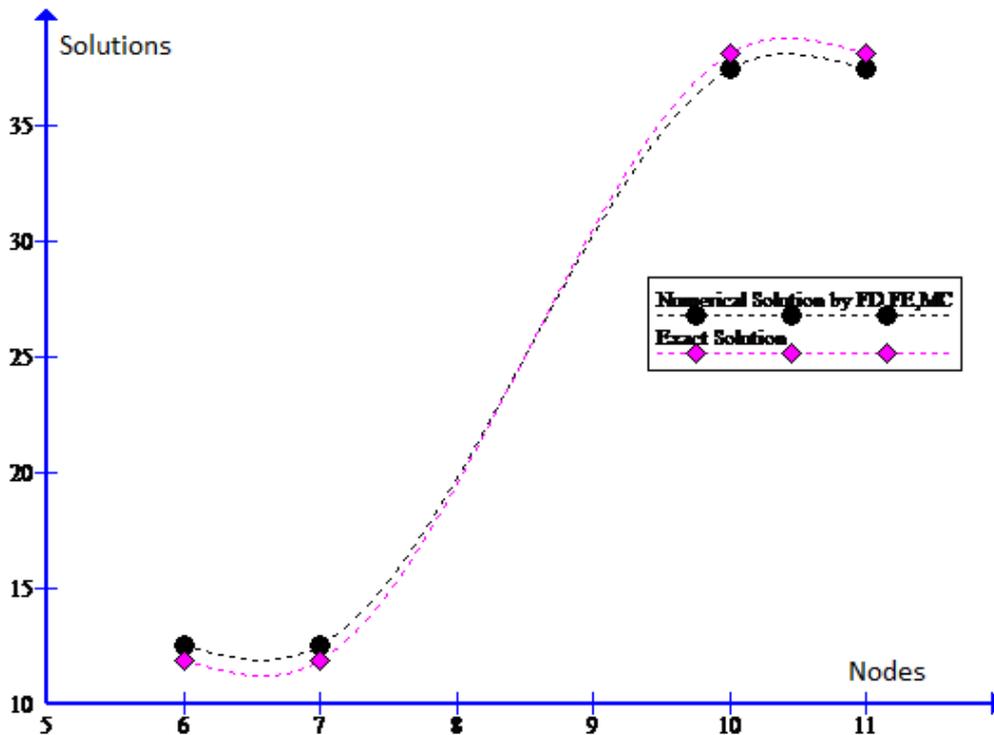


Fig.-6 Graphical Comparison between Numerical and Exact Solution

### **VIII. Conclusion**

This paper presented spreadsheet implementations of numerical methods for solving Laplace equation in two dimensions with Dirichlet boundary conditions. Other types of boundary conditions could be used to solve the same problem. By introducing a little more complexity in formulating the Finite Element Method, many useful and practical problems could be solved. The power of the Finite Element method becomes more evident, because the Finite Difference method will have much more difficulty in solving problems in a domain with complex geometries. It is observed that Finite difference, Finite element and Markov Chain Solution give near to the exact solution.

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