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On Spaces of Entire Functions Having Slow Growth Represented By Dirichlet Series

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Abstract: *In this paper spaces of entire function represented by Dirichlet Series have been considered. A norm has been introduced and a metric has been defined. Properties of this space and a characterization of continuous linear functionals have been established.*

1. Let,

(1.1) $f(s) = \sum_{n=1}^{\infty} a_n \,.\, e^{s.\lambda_n}$.

where $0 < \lambda_1 < \lambda_2 < \ldots\ldots\ldots < \lambda_n \ldots\ldots\ldots, \lambda_n \to \infty$ as $n \to \infty$, $s = \sigma + it$ (σ, t being reals) and $\{a_n\}_1^{\infty}$ any sequence of complex numbers, be a Dirichlet Series. Further, Let -:

(1.2) $\lim_{\sigma\to\infty} \sup\ n/\lambda_n = D < \infty$.

(1.3) $\lim_{\sigma\to\infty} \sup\ (\lambda_{n+1} - \lambda_n)$.

and

(1.4) $\lim_{\sigma\to\infty} \sup \dfrac{\log |a_n|^{-1}}{\lambda_n}$

Then the series in (1.1) represents an entire function f(s). We donate by X the set of all entire functions f(s) having representation (1.1) and satisfying the conditions (1.2)-(1.4). By giving different topologies on the set X, Kamthan[4] and Hussain and Kamthan [2] have studied various topological properties of these spaces. Hence we define, for any non- decreasing sequence $\{r_i\}$ of positive numbers, $r_i \to \infty$,

(1.5) $\| f \|_{r_i} = \Sigma |a_n| e^{r_i \lambda_n}$, i=1, 2, 3 .... where $f \in X$. Then from (1.4), $\| f \|_{r_i}$ exists for each i and is a form of X. Further, $\| f \|_{r_i} \leq \| f \|_{r_{i+1}}$. With these countable numbers of norms, a metric d is defined on X as :

(1.6) $d(f, g) = \sum_{i=1}^{\infty} 1/2_i \dfrac{\| f-g \|_{r_i}}{1 + \| f-g \|_{r_i}}$

Further, following functions are defined for each $f \in X$, namely

(1.7) $p(f) = \sup_{n \geq 1} | a_n |^{1/\lambda_n}$.

(1.8) $\| f \|_{r_i} = \sup_{n \geq 1} | a_n |^{1/\lambda_n}$.

Then p (f) and $\| f \|_i$ are para-norms on X. Let

(1.9) $s(f, g) = \sum_{i=1}^{\infty} 1/2_i \dfrac{\| f-g \|_{r_i}}{1+\| f-g \|_{r_i}}$, $f, g \in X$.

It was shown [2, Lemma 1] that the three topologies induced by d, s and p on X are equivalent. Many other properties of these spaces were also obtained (see [2], pp. 206-209).
For the space of entire functions of finite Ritt order [6] and type, yet another norm $\| f \|_q$ and hence metric λ was introduced and the properties of this space X_λ were studied.

Let, for f ϵ X,

$$M(\sigma, t) \equiv M(\sigma) = \sup_{-\infty < t < \infty} | f_\mu(\sigma + it) |$$

Then M (σ) is called the maximum modulus of f(s). The Ritt order of f(s) is defined as

(1.10) $$\lim_{\sigma\to\infty} \sup \frac{\log\log M(\sigma)}{\sigma} = \rho,\ 0 \le \rho \le \infty.$$

For f < ∞, the entire function f is said to be of finite order. A function f(σ) is said to be proximate order [3] if

(1.11) $\rho(\sigma) \to \rho$ as $\sigma \to \infty$, $0 < \rho < \infty$,

(1.12) $\sigma \rho'(\sigma) \to 0$ as $\sigma \to \infty$.

For f ϵ X, define

(1.13) $$\lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{e^{\sigma\rho(\sigma)}} \le A < \infty.$$

Then it was proved [3] that (1.13) holds if and only if

(1.14) $\lim_{n\to\infty} \sup \phi(\lambda_n) |a_n|^{1/\lambda_n} < (A.e\,\rho)^{1/\rho}$, where $\phi(t)$ is the unique solution of the equation t = exp $[\sigma.\rho(\sigma)]$.

(Apparently the inequality (4.1) and the definition of $\phi(t)$ contain some misprints in [2, pp.209-210]).
For each f ϵ X, define

$$\| f \|_q = \sum_{n=1}^{\infty} |a_n| \left\{ \frac{\phi(\lambda_n)}{[(A+1/q)e]^{1/\rho}} \right\},$$

Where q = 1, 2, 3............ . For $q_1 \le q_2$, $\| f \|_{q1} \le \| f \|_{q2}$. It was proved that $\| f \|_q$, q = 1, 2, 3............ Induces on X a unique topology such that X becomes a convex topological vector space, where this topology is given by the metric λ,
this space was donated by X_λ. Various properties of this space were studied [2, pp. 209-216].

It is evident that if ρ = 0, then the definition of the norm $\| f \|_q$ and proximate order ρ (σ) is not possible. It is the aim of this paper to give a metric on the space of entire functions of zero order thereby studying some properties of this space.

2. For an entire function f(s) represented by (1.1), for which ρ defined by (1.10) is equal to zero, we define following Rahman [5].

(2.1) $$\lim_{\sigma\to\infty} \sup \frac{\log\log M(\sigma)}{\log \sigma} = \rho^*,\ 1 \le \rho^* \le \infty.$$

Then ρ* is said to be the logarithmic order of f(s). For 1 < ρ* < ∞, we define the logarithmic proximate order [1] ρ*(σ) as a continuous piecewise differentiable function for $\sigma > \sigma_0$ such that

(2.2) $\rho^*(\sigma) \to \rho^*$ as $\sigma \to \infty$,
(2.3) $\sigma.\log \sigma.\ \rho^*(\sigma) \to 0$ as $\sigma \to \infty$.

Then the logarithmic type T* of {f} with respect to proximate order ρ*(σ) is defined as [7]:

(2.4) $$\lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{\sigma\rho^*(\sigma)} = T^*,\ 0 < T^* < \infty.$$

It was proved by one of the authors [7] that f(s) is of logarithmic order ρ^*, $1 < \rho^* < \infty$ and logarithmic type T^*, $0 < T^* < \infty$ if and only if

$$(2.5) \quad \lim_{n\to\infty} \sup \frac{\lambda\phi(\lambda_n)}{\log |a_n|^{-1}} = \frac{\rho^*}{\rho^*-1} (\rho^* T^*)^{1/(\rho^*-1)},$$

where $\phi(t)$ is the unique solution of the equation $t = \sigma^{\rho^*(\sigma)-1}$. We now denote by X the set of all entire functions f(s) given by (1.1), satisfying (1.2) to (1.4), for which

$$(2.6) \quad \lim_{\sigma\to\infty} \sup \frac{\log M(\sigma)}{\sigma\rho^*(\sigma)} \leq T^* < \infty,\ 1 < \rho^* < \infty.$$

Then from (2.5), we have

$$(2.7) \quad \lim_{n\to\infty} \sup \frac{\lambda_n\phi(\lambda_n)}{\log |a_n|^{-1}} \leq \frac{\rho^*}{\rho^*-1} (\rho^* T^*)^{1/(\rho^*-1)}.$$

In all our further discussion, we shall denote $(\rho^*/\rho^*-1)^{(\rho^*-1)}$ by the constant K. Then from (2.7) we have

$$(2.8) \quad |a_n| < \exp\left[-\frac{\lambda_n.\phi(\lambda_n)}{\{K.\rho^*(T^*+\epsilon)\}^{1/(\rho^*-1)}}\right],$$

Where $\epsilon > 0$ is arbitrary and $n > n_0$. Now for each $f \in X$, let us define

$$\|f\|_q = \sum_{n=1}^{\infty} (|a_n|) \exp\left[-\frac{\lambda_n.\phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}}\right],$$

Where q = 1, 2, 3, ... In view of (2.8), $\|f\|_q$ exists and for $q_1 \leq q_2$, $\|f\|_{q_1} \leq \|f\|_{q_2}$. This norm induces a metric topology on X.

We define

$$\lambda(f, g) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\|f-g\|_q}{1 + \|f-g\|_q}.$$

The space X with the above metric λ will be donated by X_λ. Now we prove

Theorem 1 –: The space X_λ is a Fre'chet space.

Proof –: It is sufficient to show that X_λ is complete. Hence, let $\{f_\alpha\}$ be a λ-Cauchy sequence in X. Therefore, for any given $\epsilon > 0$ there exists $n_0 = n_0(\epsilon)$ such that –:

$$\|f_\alpha - f_\beta\|_q < \epsilon \ \forall\ \alpha, \beta > n_0,\ q \geq 1.$$

Denoting $f_\alpha(s) = \sum_1^\infty a_n(\alpha)\, e^{s.\lambda n}$, $f_\beta(s) = \sum_1^\infty a_n(\beta)\, e^{s.\lambda n}$, we have therefore

$$(2.9) \quad \sum_1^\infty (|a_n(\alpha) - a_n(\beta)|).\exp\left[\frac{\lambda_n.\phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}}\right] < \epsilon$$

For $\alpha, \beta > n_0$, $q \geq 1$. Hence we obviously have

$|a_n(\alpha) - a_n(\beta)| < \epsilon\ \forall\ \alpha, \beta > n_0$, i.e. $\{a_n(\alpha)\}$ is a Cauchy sequence of

Complex Numbers for each fixed n = 1, 2, 3 ....

Now letting $\beta \to \infty$ in (2.9), we have for $\alpha > n_0$,

$$(2.10)\ \sum_{n=1}^{\infty} \left(| a_n(\alpha) - a_n | \right). \exp \left[- \frac{\lambda_n.\Phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}} \right] < \epsilon.$$

Taking $\alpha = n_0$, we get for a fixed q,

$$| a_n |.\exp \left[\frac{\lambda_n.\Phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}} \right] \leq | a_n^{(n_0)} |.\exp \left[\frac{\lambda_n.\Phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}} \right] + \epsilon.$$

Now, $f^{(n_0)} = \sum_{n=1}^{\infty} a_n^{(n_0)} . e^{s.\lambda_n} \in X_\alpha$, hence the condition (2.8) is satisfied. For arbitrary p > q, we have

$$| a_n^{(n_0)} | < \exp \left[- \frac{\lambda_n.\Phi(\lambda_n)}{\{K\rho^*(T^*+1/p)\}^{1/(\rho^*-1)}} \right] \text{ for arbitrarily large n.}$$

Hence we have

$$| a_n | \exp\left[- \frac{\lambda_n.\Phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}} \right] < \exp\left[- \lambda_n.\Phi(\lambda_n) \frac{1}{\{K\rho\}^{1/(\rho^*-1)}} \left[\frac{1}{(T^*+1/q)\}^{1/(\rho^*-1)}} - \frac{1}{(T^*+1/p)\}^{1/(\rho^*-1)}} \right] \right] + \epsilon$$

Since $\epsilon > 0$ is arbitrary and the first term on the R.H.S. $\to 0$ as $n \to \infty$, we find that the sequence $\{a_n\}$ satisfies (2.8). Then $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ belongs to X_λ.

Now, from (2.10), we have for q = 1, 2, 3 ...., $\| f_\alpha - f \|_q < \epsilon$. Hence,

$$\lambda (f_\alpha , f) = \sum_{q=1}^{\infty} \frac{1}{2^q} \frac{\| f_\alpha - f \|_q}{1 + \| f_\alpha - f \|_q} \leq \frac{\epsilon}{(1+\epsilon)} \sum_{q=1}^{\infty} \frac{1}{2^q} = \frac{\epsilon}{(1+\epsilon)} < \epsilon.$$

Since the above inequality holds for all $\alpha > n_0$, we finally get $f_\alpha \to f$ where $f \in X_\lambda$. Hence X_λ is complete. This proves theorem 1.

Theorem 2 –: A continuous linear functional ψ on X_λ is of the form

$$\psi (f) = \sum_{n=1}^{\infty} a_n c_n \text{ if and only if}$$

$$(2.11)\ | c_n | \leq A. \exp \left[\frac{\lambda_n.\Phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}} \right]$$

For all $n \geq 1$, $q \geq 1$, where A is a finite, positive number, $f = f(s) = \sum_{n=1}^{\infty} a_n . e^{s.\lambda_n}$ and λ_1 is sufficiently large.

Proof –: Let $\psi \in X'_\lambda$. Then for any sequence $\{ f_m \} \in X_\lambda$ such that $f_m \to f$, we have $\psi(f_m) \to \psi(f)$ as $m \to \infty$. Now let –:

$$f(s) = \sum_{n=1}^{\infty} a_n . e^{s.\lambda_n} ,$$

where a'_ns satisfy (2.8). Then $f \in X_\lambda$. Also, let

$$f_m(s) = \sum_{n=1}^{m} a_n . e^{s.\lambda_n}$$

Then $f_m \in X_\lambda$ for m = 1, 2, 3 .... Let q be any fixed positive integer and let $0 < \epsilon < 1/q$. From (2.8), we can find an integer m such that

$$| a_n | < \exp \left[- \frac{\lambda_n.\Phi(\lambda_n)}{\{K\rho^*(T^*+\epsilon)\}^{1/(\rho^*-1)}} \right] , n > m$$

Then,

$$\| f - \sum_{n=1}^{m} a_n . e^{s.\lambda_n} \|_q = \| \sum_{n=m+1}^{\infty} a_n . e^{s.\lambda_n} \|_q$$

$$= \sum_{n=1+m}^{\infty} (| a_n |) \exp \left[\frac{\lambda_n . \Phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}} \right]$$

$$< \sum_{n=1+m}^{\infty} \exp \left[\frac{\lambda_n . \Phi(\lambda_n)}{\{K\rho^*\}^{1/(\rho^*-1)}} \right] \left[\frac{1}{(T^*+1/q)\}^{1/(\rho^*-1)}} - \frac{1}{(T^*+1/p)\}^{1/(\rho^*-1)}} \right]$$

$< \epsilon$ for sufficiently large values of m.

Hence, $\lambda(f, f_m) = \sum_{q=1}^{\infty} 1/2^q \dfrac{\| f-f_m \|_q}{1 + \| f-f_m \|_q} \leq \dfrac{\epsilon}{(1 + \epsilon)} < \epsilon.$

i.e. $f_m \rightarrow f$ as $m \rightarrow \infty$ in X_λ. Hence by assumption that $\psi \in X'_\lambda$, we have

$$\lim_{m\rightarrow\infty} \psi(f_m) = \psi(f).$$

Let us donate by $C_n = \psi(e^{s.\lambda_n})$. Then

$$\psi(f_m) = \sum_{n=1}^{m} a_n \psi(e^{s.\lambda_n}) = \sum_{n=1}^{m} a_n C_n$$

Also $| C_n | = | \psi(e^{s.\lambda_n}) |$. Since ψ is continuous on X_λ, it is continuous on $\| X \|_q$ for each q = 1, 2, 3 .... Hence there exists a positive constant A independent of q such that

$$| \psi(e^{s.\lambda_n}) | = | C_n | \leq A \| \alpha \|_q , q \geq 1,$$

where $\alpha(s) = e^{s.\lambda_n}$. Now using the definition of the norm for $\alpha(s)$, we get

$$| C_n | \leq A.\exp \left[\frac{\lambda_n . \Phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}} \right], n \geq 1, q \geq 1.$$

Hence we get $\psi(f) = \sum_{n=1}^{\infty} a_n C_n$, where C'_ns satisfy (2.11).

Conversely, suppose that $\psi(f) = \sum_{n=1}^{\infty} a_n C_n$ and C_n satisfies (2.11). Then for $q \geq 1$,

$$| \psi(f) | \leq A . \sum_{n=1}^{\infty} (| a_n |) \exp \left[\frac{\lambda_n . \Phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}} \right],$$

i.e. $| \psi(f) | \leq A. \| f \|_q , q \geq 1$,

i.e. $\psi \in \| X' \|_q , q \geq 1$. Now, since

$$\lambda(f, g) = \sum_{q=1}^{\infty} 1/2^q \frac{\| f-f_m \|_q}{1 + \| f-f_m \|_q},$$

therefore $X'_\lambda = \bigcup_{q=1}^{\infty} \| X' \|_q$. Hence $\psi \in X'_\lambda$.

This completes the proof of Theorem 2. Lastly, we give the construction of total sets in X_λ. Following [2], we give –:

Definition –: Let X be a locally convex topological vector space A set $E \subset X$ is said to be total if and only if for any $\psi \in X'$ with $\psi(E) = 0$, we have $\psi = 0$.

Now, we prove

Theorem 3 –: Consider the space X_λ defined before and let $f(s) = \sum_{n=1}^{\infty} a_n (e^{s.\lambda_n})$,

$a_n \neq 0$ for n = 1, 2, 3 .... , f ϵ X_λ. Suppose G is a subset of the complex plane having at least one limit point in the complex plane. Define, for μ ϵ G, $f_\mu(s) = \sum_{n=1}^{\infty} (a_n e^{\mu.\lambda} n).(e^{s.\lambda} n)$. Then E = { f_μ ; μ ϵ G } is total in X_λ .

Proof –: Since f ϵ X_λ , from (2.7) we have

$$\lim_{n\to\infty} \sup \frac{\lambda_n.\phi(\lambda_n)}{\log | a_n.e^{\mu.\lambda} n |^{-1}} = \lim_{n\to\infty} \sup \frac{\phi(\lambda_n)}{\log | a_n |^{-1/\lambda} n - R(\mu)}$$

$$\leq \frac{\rho^*}{\rho^* - 1} (\rho^* T^*)^{1/(\rho^*-1)} \text{, since } R(\mu) < \infty.$$

Hence, if we donate by $M_\mu(\sigma) = \sup_{-\infty < t < \infty} | f_\mu (\sigma + it) |$, then from (2.6),

$$\lim_{\sigma\to\infty} \sup \frac{\log M_\mu(\sigma)}{P^*(\sigma)} \leq T^* < \infty.$$

Therefore, f_μ ϵ X_λ for each μ ϵ G. Thus E ⊂ X_λ . Now, let ψ be a linear continuous functional on X_λ and suppose that $\psi(f_\mu) = 0$. From Theorem 2, there exists a sequence {C_n} of complex numbers such that

$$\psi(g) = \sum_{n=1}^{\infty} b_n C_n , \ g(s) = \sum_{n=1}^{\infty} b_n e^{s.\lambda} n \in X_\lambda ,$$

where

$$\text{(2.12) } | C_n | < A.\exp \left[\frac{\lambda_n.\phi(\lambda_n)}{\{K\rho^*(T^*+1/q)\}^{1/(\rho^*-1)}} \right] , \ n \geq 1, \ q \geq 1,$$

A being a constant and λ_1 is sufficiently large.
Hence,

$$\psi(f_\mu) \sum_{n=1}^{\infty} a_n C_n e^{\mu\lambda} n = 0 , \ \mu \in G.$$

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