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Cash Flow Valuation Mode Lin Discrete Time

Olayiwola. M. A. and Oni, N. O.
Department of Mathematics and Statistics, The Polytechnic, Ibadan, Saki Campus.

Abstract: *This research consider the modelling of each cash flow valuation in discrete time. It is shown that the value of cash flow can be modeled in three equivalent ways under same general assumptions. Also, consideration is given to value process at a stopping time and/ or the cash flow process stopped at some stopping times.*

I. Introduction

It is observed that money has a time value that is to value an amount of money we get at some future date we should discount the amount from the future date back to today.

In finance and economics, discounted cash flow analysis is a method of valuing a project, company, or asset using the concept of the time value of money. All future cash flows are estimated and discounted to give their present values. The sum of all future cash flows, both incoming and outgoing, is the net present value, which is taken as the value or price of the cash flows in question.

Most future cash flows models in finance and economics are assumed to be stochastic. Thus, to value these stochastic cash flow we need to take expectations.

1.1 Probability Space

Probability theory is concerned with the study of experiments whose outcomes are random; that is the outcomes cannot be predicted with certainty. The collection Ω of all possible outcomes of a random experiment is called a sample space. Therefore, an element of ω of Ω is called a sample point.

Definition 1: A collection ฿ofΩis called a σ-algebra if
(i) $\Omega \epsilon$ ฿
(ii) $\Omega \epsilon$ ฿, then$A^C \epsilon$ ฿
(iii) For any countable collection $\{A_n, n \geq 1\} \subset$ ฿ we have $U_n \geq 1, A_n \epsilon$ ฿
Definition 2: Let Ω be a sample space and ฿ is a σ $-$algebra of subsets of Ω. A function $P(\cdot)$ defined on ฿ and taking values in the unit interval$[0,1]$ is called a probability measure if
(i) $P(\Omega) = 1$
(ii) $P(A) \geq 0 \;\; \forall \, A \, \epsilon$ ฿
(iii) for at most countable family $A_n, n \geq 1$, of mutually disjoint events we have

$$P\{U_{n\geq 1}, A_n\} = \sum_{n\geq 1} P(A_n)$$

The triple (Ω,฿, $\mathbb{P}$) is called probability space. The pair (Ω,฿)is called the probabilizable space.

Definition 3: A probability space (Ω,฿, $\mathbb{P}$)is called a complete probability space if every subset of a $\mathbb{P}$- null event is also an event.

1.1.1 **Filtered Probability Space**

Let (Ω,฿, $\mathbb{P}$)be a probability space and F(฿) = {฿$_t$ $= t\epsilon[o,\infty]\}$, be a family of sub σ $-$algebraof ฿such that

(i) ฿contained all the $\mathbb{P}$– null member of ฿
(ii) ฿$_s \subseteq$ ฿$_t$whenever $t > s \geq 0$ where ฿$_s$ is sub σ $-$algebra of ฿$_t$
(iii) F (฿) is right continuous in the sense that ฿$_{t+}$ = ฿$_t$, $t \geq 0$ where ฿$_{s>t}$ =∩ ฿$_{s>0}$ then F (฿) = {฿$_t$$: t\epsilon[o,\infty]\}$is called filtration of ฿.

The quadruplet(Ω,฿, $\mathbb{P}$,F (฿)) is called a filter probability space.

If F(฿) = {฿$_t$$: t\epsilon[o,\infty]\}$ is a filtration of ฿, then ฿$_t$ is the information available about ฿ at time t while filtration F (฿) describes the flow of information with time.

Definition 4: Let (Ω,฿, ℙ,F(฿))be a filtered probability space with the filtration express as F (฿) = {$฿_t : t\epsilon[o,\infty]$} and let $X(\pi) = \{X(t), t \,\epsilon\, \pi\}$ be a stochastic process with values in R, then X(t) is called an adapted to the filtration F(฿)if X(t) is $฿_t$measurable for $t \,\epsilon\, \pi$. Where X(t) is a random variable.

1.2 **Mathematical Expectation**

Let $X = e^0(\Omega\,,\, R^d)$, then the mean or mathematical expectation of X is defined by

$$E(X) = \int_{\Omega} X(\omega)\mu(d\omega) \quad \omega \,\epsilon\, \Omega$$

If X has probability density function f_x which is absolutely continuous with respect to lebesgue measure then

$$E(X) = \int_{R^d} x f_x(x) dx$$

For a random variable $X \;= (X_1, X_2, . \;. \;. X_n)$

$$E(X) = E(X_1, X_2, \ldots X_d\,)$$
$$= E(X_1) \,\cdot E(X_2) \,\cdots E(X_d)$$

1.2.1 **Conditional Expectation**

Let X_1 and X_2 be two random variables, then the conditional expectation of X_1 given that $X_2 = x_2$ is defined as

$$E(X_1|X_2 = x_2) = \; E(X_1|x_2)$$
$$= \int x_1 f(x_1|x_2)\, dx_1$$
$$= \int x_1 \frac{f(x_1, x_2)}{f(x_2)}\, dx_1 \qquad f(x_2) > 0$$

Properties of Conditional Expectation

(i) Let $a\,, b\,\epsilon\, R\,,\; X, Y\; are\; IR - valued\; random\; variables\; and$ F is a sub σ −algebra of ฿ then

$E[(aX + bY)|$F $] \;=\; E(aX|$F) + E(bY|F)

$= \; a\,E(X|$F) + $b\,E(\,Y|$F)

(ii) If $X \geq 0\;\; a.s\,, then\; E(X|$F)≥ 0 $\;a.s\; E(\cdot\,|$F) is positively preserving.

(iii) E(C|F) = C where C is a constant value.

(iv) If X is F - measurable, then E(X|F) = X

(v) If X is F - measurable, then E(X|F) = E(X)

(vi) If F_1 and F_2 are both σ −algebra of฿ such that $F_1 \subset F_2$ then

$E(E(X|F_2|F_1)$ = E(X|F_1) a.s

(vii) If $X\,\epsilon\, \mathcal{L}^I$(Ω,฿, ℙ) and $X_n \rightarrow\; X_1$then$E(X_n|$ F)→ $E(X|$ F) in $\mathcal{L}^I$(Ω,฿, ℙ)

(viii) If φ: $R \rightarrow R$ is convex and φ(x)$\epsilon\; \mathcal{L}^I$(Ω,฿, ℙ)such that

E|φ(x)|< ∞ then E (φ(x)|F)≥ φE(X|F) a.s

(xi) If X is independent of F_1then E(X|F) = E(X)

1.3 **Stochastic Processes**

Definition 5: Let $\{X_t(\omega),\; \omega\,\epsilon\,\Omega\,,\; t\epsilon[o,\infty]\}$ be a family of R^d – valued random variable define on the probability space (Ω,F, ℙ) then the family$\{X(t,\omega)t\epsilon[o,\infty]\}$of X is called a stochastic process $\omega\,\epsilon\,\Omega$

A stochastic process $\{X_t(t,\omega),\; \omega\,\epsilon\,\Omega\,,\; t\,\epsilon\,\pi\}$ is thus a function of two variables

X: T X Ω $\rightarrow R^d$, $\{X_t\}\, for\; t \geq 0$ is said to be F_t – adapted if for every $t \geq 0$, X_t is F_t measurable.

1.4 **Martingale Process**

Definition 6: A stochastic process $\{X_n\}_{n\geq 1}$ is said to be a Martingale process if

(i) $E[X \mid X_n] \;<\; \infty$

(ii) $E[X_{n+1} \mid X_1\,, X_2\,, \ldots\; X_n] \;=\; X_n$

Definition 7: Let $X(t.\pi) = \{X(t), t\,\epsilon\,\pi\}$ be an adapted R – valued stochastic process on a filtered probability space (Ω,F, $(F_t)_t\,\epsilon\pi$,P) then X is called martingale if for each $t\,\epsilon\,\pi \subseteq$ R E[X(t)|F] = X(s) a.s s < t.

II. Model Formulation

The discounted cash flow formular is derived from the future value formular for computing the time value of money and compounding returns.

$$C = V(1 + r)^n$$

Discounted present value (DPV) for one cash flow (FV) in one future period is expressed as

$$V = \frac{C}{(1+r)^n} = FV(1-d)^n$$

Where,

- V is the discounted present value;
- C is the norminal value of a cash flow amount in a future periods;
- r is the interest rate, which reflects the cost of tying up capital and may also allow for the risk that the payment may not be received in full;
- d is the discount rate, which is $\frac{r}{(1+r)}$
- n is the time in years before the future cash flow occurs.

For multiple cash flows in multiple time periods to be discounted it is necessary to sum them and take the expectation.

$$V_0 = E\left[\sum_{k=1}^{\infty} \frac{C}{(1+r)^k}\right] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

We take the expectation because we assume that most cash flows models are stochastic.
Introduction of the value at time $t \geq 0$ we have a dynamic model

$$V_t = E_t\left[\sum_{k=t+1}^{\infty} \frac{C_k}{(1+r)^{k-t}}\right]$$

Here $E_t(\cdot)$ is the expectation given information up to and including time t.

$$letting \qquad m_k = \frac{1}{(1+r)^k}$$

$$V_0 = E\left[\sum_{k=1}^{\infty} C_k m_k\right]$$

2.1 **General definitions**

Definition 2.1 : A cash flow $(C_t)\; t \in N$ is a process adapted to the filtration F_t and that for each $t \in N,\; C_t < \infty \; a.s$

A cash flow process that is non-negative a.s will be referred to as a dividend process.

Definition 2.2 A discount process is a process $m: N \times N \times \Omega \to R$ satisfying

(i) $0 < m(s,t) < \infty \;\; a.s \; \forall \; s,t \in N$

(ii) $m(s,t,\omega)\; is\; F_{max(s,t)}$ – measurable $\forall \; s,t \in N \; and$

(iii) $m(s,t) = m(s,u)m(u,t)\; a.s \; \forall \; s,u,t \in N$ A discount process fulfilling

$i^l \;\; 0 < m(s,t) < 1 \; a.s \; \forall \; s,t \in N \; with \; s \leq t$will be referred to as normal discount process

$m(s,t)$is interpreted as the stochastic value at time s of getting one unit of currency at time t, while $m(t,s)$will be referred to as the growth of one unit of currency invested at time t, at time s. We write $m_t = m(0,t),\; t \in N$ as a short-hand notation.

Definition 2.3 The discount rate or instantaneous rate at time t implied by discount process denoted by r_t for t = 1, 2, . . . is defined as

$$r_t = \frac{1}{m(t-1,t)} - 1 = \frac{\Lambda(t-1)}{\Lambda(t)} - 1$$

Where $\wedge$ is the deflator associated with m. The relationship between the discount process and rate process is deducted from the following Lemma.

Lemma 1: Let m be a discount process and let r be the discount rate implied by m. Then the following holds:

(i) $-1 < r_t < \infty, \qquad t \in \mathbb{N}$

(ii) $r \geq 0 \;\; iff \; m \; is \; normal \; discount \; process$

(iii) For any given $\boldsymbol{\lambda} \geq \mathbf{0}$ we have for $t \in \mathbb{N}$

$$0 < \boldsymbol{\lambda} < \boldsymbol{r_t} \quad \boldsymbol{iff}$$
$$0 < m_t \leq e^{-tln\,(1+\lambda)}$$

(iv) The instantaneous rate process and discount process uniquely determine each other.

III. Valuation of Cash Flow

Definition 3: Given a cash flow process C_t for $t \in \mathbb{N}$ and a discount process $m(s,t)$ for $s,t \in N$ we define for $t \in \mathbb{N}$ the value process as

$$V_t = E[\textstyle\sum_{k=t+1}^{\infty} C_k m(t,k) | F_t\,]$$

The value process is defined as ex – dividend if cash flows from time t+1 and onwards is included in the value at time t. It is otherwise considered as cum – divided if the cash flow is included at time t.
The following lemma gives a sufficient condition as to when is the value process will be finite a.s
Lemma 2: If C is a cash flow process and $E[\,|\sum_{k=1}^{\infty} C_k m_k\,|] < \infty \;\; a.s$
Then $|V_t| < \infty$ a.s for all $t \in \mathbb{N}$
Proof:
Since $E[\,|\sum_{k=1}^{\infty} C_k m_k\,|] < \infty$ for $t \in \mathbb{N}$
$|V_t| = E[|\sum_{k=t+1}^{\infty} C_k m(t,k)|F_t\,] = E[|\sum_{k=t+1}^{\infty} C_k m(t,0)m(0,k)|F_t\,]$
$\leq m(t,0)E[|\sum_{k=t+1}^{\infty} C_k m(0,k)|F_t\,]$
$\leq \frac{1}{m(0,t)} E[|\sum_{k=t+1}^{\infty} C_k m_k|F_t\,]$
$\leq \frac{1}{mt} E[|\sum_{k=1}^{\infty} C_k m_k + \sum_{k=t+1}^{\infty} C_k m_k - \sum_{k=1}^{t} C_k m_k|F_t\,]$
$\leq \frac{1}{mt} E[|\sum_{k=1}^{\infty} C_k m_k|F_t + |\sum_{k=1}^{t} C_k m_k|] < \infty \qquad a.s$
Thus, if C is a discount process such that $V_0 < \infty$, then $V_t < \infty$ a.s for every $t \in \mathbb{N}$
Note that if X is a random variable with $E|X| < \infty$ then $E[X|F_t]$ for t = 1, 2, . . . , is a uniformly integrable (U.I) martingale. Hence, if $[|\sum_{k=1}^{\infty} C_k m_k|] < \infty$ a.s then $E[\sum_{k=0}^{\infty} C_k m_k|F_t\,]$ is a U.I martingale.
Below proposition gives the characteristics of value process
Proposition 2: Let C and m be a cash flow and discount process respectively. If $[|\sum_{k=1}^{\infty} C_k m_k|] < \infty$ then the discounted value process $(V_t\, m_t)$ can be written as

$$V_t\, m_t = M_t - A_t \quad for \quad t \in \mathbb{N}$$

Where M is a U.I martingale and A an adopted process. Furthermore,

$$\lim_{t\to\infty} V_t\, m_t = 0 \qquad a.s$$

Proof:
Note that $[|\sum_{k=0}^{\infty} C_k m_k|] < \infty \;\; a.s \;\; since E \quad [|\sum_{k=0}^{\infty} C_k m_k|] < \infty$
$Let \quad M_t = E[\sum_{k=0}^{\infty} C_k m_k|F_t]$ for $t \in \mathbb{N}$

$$A_t = \sum_{k=0}^{t} C_k m_k \;, \qquad\qquad t \in \mathbb{N}$$

It is then immediate that $V_t\, m_t = M_t - A_t$. Since $[|\sum_{k=1}^{\infty} C_k m_k|] < \infty$, M is a U.I martingale and A ia adapted.
As $t \to \infty$, the UI martingale
$M_t = E[\sum_{k=0}^{\infty} C_k m_k|F_t] = \sum_{k=0}^{\infty} C_k m_k = A_\infty \quad a.s$
This implies that as $t \to \infty$

$$\lim_{t\to\infty} V_t\, m_t = \lim_{t\to\infty} M_t = \lim_{t\to\infty} A_t = 0$$

Since $M_\infty = A_\infty = \sum_{k=0}^{\infty} C_k m_k$ is finite a.s . Now let C be dividend process.
Then A is an increasing process and $E[A_\infty] = E[\sum_{k=0}^{\infty} C_k m_k] < \infty$ by assumption. Thus:
$V_t\, m_t = E[\sum_{k=t+1}^{\infty} C_k m_k|F_t] = E[A_\infty|F_t\,] - A_t$
Characteristics relation between C (cash flow process), m(discount process) and V(value process) is given in the following theorem.
Theorem 3: Let C be a cash flow process and m a discount process such that $E[|C_k|m_k] < \infty$. Then the following three statements are equivalent.
(i) For every $t \in \mathbb{N}$
$V_t = E[\sum_{k=t+1}^{\infty} C_k m(t,k)|F_t]$

(ii) For every $t \in \mathbb{N}$

$$M_t = V_t\, m_t + \sum_{k=1}^{t} C_k m_k$$

is a UI martingale
(b) $V_t\, m_t \to 0 \quad a.s \quad when \quad t \to 0$
(iii) For every $t \in \mathbb{N}$
(a) $V_t = E[m(t,t+1)(C_{t+1} + V_{t+1})|F_t]$
(b) $\lim_{T\to\infty} E[m(t,T)V_T|F_t] = 0$
Proof:
Note that $E|\sum_{k=1}^{\infty} C_k m_k| \leq E[\sum_{k=1}^{\infty} |C_k| m_k] < \infty$ which implies that $|\sum_{k=1}^{\infty} C_k m_k| < \infty \;\; a.s$ Now to show that

(i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii)

(i) $\Leftrightarrow$ (ii) The if part follows from the proposition. For the only if part the expression in (ii) a is written as

$$-m_{k+1}C_{k+1} = m_{k+1}V_{k+1} - m_k V_k - M_{k+1}M_k$$

And take the sum from t to T – 1

$$-\sum_{k=t+1}^{T} m_k C_k = m_T V_T - m_t V_t - M_T + M_t$$

If we let $T \to \infty$ the term $m_T V_T \to 0 \ a.s$ by the assumption and $M_T \to M_\infty$ a.s from the convergence result of UI martingales. Thus, we have

$$V_t m_t = \sum_{k=t+1}^{\infty} C_k m_k - M_\infty + M_t \qquad \ldots\ldots\ldots(1)$$

It is observed that $E[M_\infty|F_t] = M_t$ from the convergence result of UI martingale. Taking conditional expectation with respect to $F_t$of (1)

$E[V_t m_t|F_t] = E[\sum_{k=t+1}^{\infty} C_k m_k|F_t] - E[M_\infty|F_t] + E[M_t|F_t]$

$V_t m_t = E[\sum_{k=t+1}^{\infty} C_k m_k|F_t] - M_t + M_t$

$V_t m_t = E[\sum_{k=t+1}^{\infty} C_k m_k|F_t]$

$V_t = \frac{1}{m_t} E[\sum_{k=t+1}^{\infty} C_k m_k|F_t] = E[\sum_{k=t+1}^{\infty} C_k \frac{m_k}{m_t}|F_t]$

$= E[\sum_{k=t+1}^{\infty} C_k m(t,k)|F_t]$

Note:

(i) $M_\infty = A_\infty = \sum_{k=1}^{\infty} C_k m_k$

$E[M_\infty|F_t] = E[\sum_{k=1}^{\infty} C_k m_k|F_t] = M_t$

(ii) $E[M_t|F_t] = E[\sum_{k=1}^{\infty} C_k m_k|F_t|F_t] = E[\sum_{k=1}^{\infty} C_k m_k|F_t]$

Regarding (i) $\Leftrightarrow$ (iii)

We begin with the if part. Fix $t \in N$

$V_t = E[m(t,t+1)C_{t+1} + \sum_{k=t+2}^{\infty} C_k m(t,k)|F_t]$

$V_t = E[m(t,t+1)C_{t+1} + m(t,t+1)\sum_{k=t+2}^{\infty} C_k m(t+1,k)|F_t]$

$V_t = E[m(t,t+1)C_{t+1} + \sum_{k=t+2}^{\infty} C_k m(t+1,k)|F_t]$

$V_t = E[m(t,t+1)C_{t+1} + V_{t+1}|F_t]$

Now let $T \geq t$. *From* $V_T = \sum_{k=T+1}^{\infty} C_k m(T,K)|F_t$

We have

$V_T = E[\sum_{k=T+1}^{\infty} C_k m(T,k)m(t,k)|F_T]$

$= E[\sum_{k=T+1}^{\infty} C_k \frac{m(t,k)}{m(t,T)}|F_T]$

$V_T m(t,T) = E[\sum_{k=T+1}^{\infty} C_k m(t,k)|F_T] \ldots\ldots\ldots\ldots\ (2)$

Taking conditional expectation with respect to F_T of (2)

$E[V_T m(t,T)/F_T] = E[\sum_{k=T+1}^{\infty} C_k m(t,k)|F_T|F_T]$

$= E[\sum_{k=T+1}^{\infty} C_k m(t,k)|F_T]$

$= \frac{1}{m_t} E[\sum_{k=t+1}^{\infty} C_k m_k|F_t]$

Since $|\sum_{k=1}^{\infty} C_k m_k| \leq \sum_{k=1}^{\infty} |C_k| m_k$

And the last random variable is integrable by assumption we get for every $t \in N$ and A $\in F_t$

$$\lim_{T\to\infty} E[m(t,T)V_T 1_A] = \lim_{T\to\infty} m(t,T)V_T 1_A = 0$$

To prove the only if part we iterate (iii) a to get

$V_t = E[\sum_{k=T+1}^{T} C_k m(t,k) + m(t,T)V_T|F_t]$

$V_t = \frac{1}{m_t} E[\sum_{k=t+1}^{\infty} C_k m_k|F_t] + E[m(t,T)V_T|F_t]$

As $T \to \infty$ *lim* $E[m(t,T)V_T|F_t] \to 0$ a.s from (iii)b since

$$\left|\sum_{k=t+1}^{T} C_k m_k\right| \leq \sum_{k=1}^{\infty} |C_k| m_k$$

And the last random variable is integrable by assumption. Thus, we get

$V_T = E[\sum_{k=t+1}^{\infty} C_k m(t,k)|F_t]$ using the theorem of dominated convergence.

IV. Stopping the cash flow and value process

Theorem3: Show that the three equivalent representations of the value process at deterministic times. The theorem also assumes that cash flow stream is defined for all $t \geq 0$. In the stream, we shall consider the value process stopping time and / or the cash flow process stopped at some stopping time. Using the fact that the martingale

$M_t = V_t m_t + \sum_{k=1}^{t} C_k m_k$ from theorem 3, is uniformly integrable we can get the following results.
Proposition 4: Let C and m be cash flow process and discount process respectively such that $E[\sum_{k=1}^{\infty} |C_k| m(t,k) < \infty$ for every $t \in N$.
Further let τ *and* σ be (F_t) stopping times such that $\sigma \le \tau \ a.s$ then the following two statements are equivalent.
(i) We have
$V_\sigma = E[\sum_{k=\sigma+1}^{\sigma} C_k m(\sigma,k) + V_\tau m(\sigma,\tau) 1_{\tau<\infty} |F_\sigma]$ $\{\sigma < \infty\}$
(ii) (a) for every $t \in N$

$$M_t = V_t m_t + \sum_{k=1}^{t} C_k m_k$$

is a UI martingale, and
(b) $V_t m_t \to 0 \ a.s$ *When* $t \to \infty$
Proof: We start with (ii) $\Leftrightarrow$ (i) considering the stopping times $\tau \wedge n$
Where $n \in N$ because the stopping time τ may be unbounded we get

$$M_{\tau\wedge n} = V_{\tau\wedge n} m_{\tau\wedge n} + \sum_{k=1}^{\tau\wedge n} C_k m_k \ \ldots\ldots\ldots\ldots (3)$$

$$as \quad n \to \infty, V_{\tau\wedge n} m_{\tau\wedge n} \to V_\tau m_\tau 1_{\tau<\infty} \ a.s$$

Since $V_\tau m_\tau 1_{\tau<\infty} \to 0 \ a.s$ assuming we let $n \to \infty$ Thus, we have from (3) $M_\tau = V_\tau m_\tau 1_{\tau<\infty} + \sum_{k=1}^{\tau} C_k m_k$
Since M is uniformly integrable we take conditional expectation of M_τ
with respect to the σ −algebraF_σ to get on $\{\sigma < \infty\}$
$E[M_\tau|F_\sigma] = E[\sum_{k=1}^{\tau} C_k m_k + V_\tau m_\tau 1_{\tau<\infty}|F_\sigma]$
Note that $E[M_\tau|F_\sigma] = M_\sigma$
Thus, $M_\sigma = E[\sum_{k=1}^{\sigma} C_k m_k + \sum_{k=\sigma+1}^{\tau} C_k m_k + V_\tau m_\tau 1_{\tau<\infty}|F_\sigma]$
$M_\sigma = \sum_{k=1}^{\sigma} C_k m_k + E[\sum_{k=\sigma+1}^{\tau} C_k m_k + V_\tau m_\tau 1_{\tau<\infty}|F_\sigma]$

$$M_\sigma = E[\sum_{k=1}^{\sigma} C_k m_k + V_\sigma m_\sigma 1_{\sigma<\infty}$$

$M_\sigma = \sum_{k=1}^{\sigma} C_k m_k + E[\sum_{k=\sigma+1}^{\tau} C_k m_k + V_\tau m_\tau 1_{\tau<\infty}|F_\sigma]$
It implies
$\sum_{k=1}^{\sigma} C_k m_k + V_\sigma m_\sigma 1_{\sigma<\infty} = \sum_{k=1}^{\sigma} C_k m_k + E[\sum_{k=\sigma+1}^{\tau} C_k m_k + V_\tau m_\tau 1_{\tau<\infty}|F_\sigma]$
Hence,
$V_\sigma m_\sigma 1_{\sigma<\infty} = E[\sum_{k=\sigma+1}^{\tau} C_k m_k + V_\tau m_\tau 1_{\tau<\infty}|F_\sigma]$
$V_\sigma = E[\sum_{k=\sigma+1}^{\tau} C_k \frac{m_k}{m_\sigma} + V_\tau \frac{m_\tau}{m_\sigma} 1_{\tau<\infty}|F_\sigma]$
$V_\sigma = E[\sum_{k=\sigma+1}^{\tau} C_k m(\sigma,k) + V_\tau m(\sigma,\tau) 1_{\tau<\infty}|F_\sigma]$
This is our desired result.
To prove (i) $\Leftrightarrow$ (ii)
Let $\tau = \infty$ *and* $\sigma = t$ *for* $t \in N$ *in* (i) $i.e$
$V_\sigma = E[\sum_{k=\sigma+1}^{\tau} C_k m(\sigma,k) + V_\tau m(\sigma,\tau) 1_{\tau<\infty}|F_\sigma]$
$V_t = E[\sum_{k=t+1}^{\infty} C_k m(t,k) + V_\infty m(t,\infty)|F_t]$
$V_t = E[\sum_{k=t+1}^{\infty} C_k \frac{m_k}{m_t} + V_\infty \frac{m_\infty}{m_t}|F_t]$
$V_t m_t = E[\sum_{k=t+1}^{\infty} C_k m_k + V_\infty m_\infty|F_t]$

$$Note \quad that \quad V_t m_t \to 0 \ a.s \quad t \to \infty \ a.s$$

$V_t m_t = E[\sum_{k=t+1}^{\infty} C_k m_k|F_t]$
$V_t m_t = E[\sum_{k=1}^{t} C_k m_k + \sum_{k=t+1}^{\infty} C_k m_k - \sum_{k=1}^{t} C_k m_k|F_t]$
$V_t m_t = E[\sum_{k=1}^{t} C_k m_k - \sum_{k=1}^{t} C_k m_k|F_t]$
$V_t m_t = E[\sum_{k=1}^{\infty} C_k m_k|F_t] - \sum_{k=1}^{t} C_k m_k$
Since $E[\sum_{k=1}^{\infty} C_k m_k|F_t]$ is a UI martingale i.e M_t Thus, we have

$$V_t m_t = M_t - \sum_{k=1}^{t} C_k m_k$$

Hence, $M_t = V_t m_t + \sum_{k=1}^{t} C_k m_k$ as desired.

V. Conclusion

We have been able to show that cash flow models can be written in equivalent forms using a suitable discount factor. We proceed to consider the value process stopping at stopping time and / or cash flow process stopped at some stopping times with the use of a proposition.

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