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Numerical solution of heat equation through double interpolation

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Abstract : *In this article an attempt is made to find the solution of one-dimensional Heat equation with initial and boundary conditions using the techniques of numerical methods, and the finite differences. Applying Bender-Schmidt recurrence relation formula we found u(x ,t) values at lattice points. Further using the double interpolation we found the solution of Heat equation as double interpolating polynomial.*
Keywords - *Boundary Value Problem , Finite Difference method, Double Interpolation.*

I. INTRODUCTION

Boundary value problems occur very frequently in various fields of science and engineering such as mechanics, quantum physics, electro hydro dynamics, theory of thermal expansions. For a detailed theory and analytical discussion on boundary value problems, one may refer to Bender [1], Collatz [2], Na [3],Numerical solution of boundary value problems by splines and numerical integration are discussed by P.S. Rama Chandra Rao [4-5], Ravikanth. A.S.V. [6].

The finite difference method is one of several techniques for obtaining numerical solutions to the boundary value problems. Especially to solve partial differential equations , in which the partial derivatives are replaced by finite differences of two variables .Mortan and Mayer[7] and Cooper[8] provide a more mathematical development of finite difference methods and modern introduction to the theory of partial differential equation along with a brief coverage of numerical methods. Fletcher [9] described the method to implement finite differences to solve boundary value problems.

In this paper we describe how to solve a one-dimensional heat equation using finite difference method and double interpolation [10]. The heat equation is of fundamental importance in diverse scientific fields which describes the distribution of heat (or variation in temperature) in a given region over time. In mathematics it is the prototypical parabolic partial differential equation. In financial mathematics, the famous Black-Scholes option pricing models differential equation can be transformed in to the heat equation allowing relatively easy solution from a familiar body of mathematics. The diffusion equation, a more general version of the heat arises in connection with the study of chemical diffusion and other related processes and a direct practical application of the heat equation, in conjunction with the Fourier theory, in spherical co-ordinates, is the measurement of the thermal diffusivity in polymers. The heat equation can be efficiently solved numerically using the Bender-Schmidt method. Usually the solution of these equations is given in Fourier series form. Monte [11] applied a natural analytical approach for solving the one dimensional transient heat conduction in a composite slab.

Lu [12] gave a novel analytical method applied to the transient heat conduction equation. In this paper we found the solution of one-dimensional heat equation with certain initial and boundary conditions as a polynomial.

II. DESCRIPTION OF THE METHOD

Consider one-dimensional heat conduction equation

$$u_t = u_{xx} \tag{1}$$

Where t and x are the time and space co-ordinates respectively, in the region

$$R = [a \leq x \leq b] \times [t \geq 0]$$

With appropriate initial and boundary conditions.

If $u = u(x,t)$ is any function of two independent variables x and t then we define

$u_{rs} = f(x_r, t_s)$ where $x_r = x_0 + rh \; and \; t_s = t_0 + sk$. Here x_0 denotes the initial value of x, t_0 denotes the initial value of t, and h, k are the intervals of differencing for the variables x and t respectively. Further r, s are non negative integers.

Replacing the partial derivatives of (1) with finite differences we get Bender-Schmidt recurrence relation which is given by

$$u_{i,j+1} = u_{i,j} + \lambda\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) \tag{2}$$

Where $\lambda = \frac{k}{h^2}$ $for\ 0 \le \lambda \le \frac{1}{2}$

By using boundary conditions and equation (2) we find $u(x,t)$ values at lattice points (x_r,t_s) . Further using double interpolation that is by finding the two way differences of various orders and substituting in the general formula for double interpolation we find double interpolating polynomial of $u(x,t)$.
The following formulae are given in [10] .

2.1. Formulae

1. The operator Δ_x is defined by the equation $\Delta_x u(x,t) = u(x+h,t) - u(x,t)$ and the operator Δ_t is defined by $\Delta_t u(x,t) = u(x,t+k) - u(x,t)$.

2. Double or two way differences of various orders are defined by
$\Delta^{1+0}u_{rs} = \Delta_x u_{rs} = u_{r+1,s} - u_{rs}$
$\Delta^{0+1}u_{rs} = \Delta_t u_{rs} = u_{r,s+1} - u_{rs}$
Similarly
$\Delta^{2+0}u_{rs} = u_{r+2,s} - 2u_{r+1,s} + u_{rs}$
$\Delta^{0+2}u_{rs} = u_{r,s+2} - 2u_{r,s+1} + u_{rs}$
The general formula for the differences of different order is given by

$$\Delta^{m+n}u_{00} = \Delta^{m+0}u_{0n} - n\Delta^{m+0}u_{0,n-1} + \frac{n(n-1)}{2}\Delta^{m+0}u_{0,n-2} + \ldots\ldots + (-1)^n\Delta^{m+0}u_{00} \tag{3}$$

$$\Delta^{n+m}u_{00} = \Delta^{0+n}u_{m0} - m\Delta^{0+n}u_{m-1,0} + \frac{m(m-1)}{2}\Delta^{0+n}u_{m-2,0} + \ldots\ldots + (-1)^m\Delta^{0+n}u_{00}$$

Where m,n are positive integers.
General formula for double interpolation is

$$u(x,t) = u_{00} + \left(\frac{(x-x_0)}{h}\Delta^{1+0}u_{00} + \frac{(t-t_0)}{k}\Delta^{0+1}u_{00}\right) + \frac{1}{2!}\left(\frac{((x-x_0)(x-x_1))}{h^2}\Delta^{2+0}u_{00} + \frac{2((x-x_0)(t-t_0))}{hk}\Delta^{1+1}u_{00} + \frac{((t-t_0)(t-t_1))}{k^2}\Delta^{0+2}u_{00})\right) + \cdots\ldots\ldots\ldots + \frac{1}{m!}\left(\frac{(x-x_0)(x-x_1)\ldots..(x-x_{m-1})}{h^m}\Delta^{m+0}u_{00} + \frac{m(x-x_0)(x-x_1)\ldots..(x-x_{m-2})(t-t_0)}{h^{m-1}k}\Delta^{(m-1)+1}u_{00} + \ldots + \frac{(t-t_0)(t-t_1)\ldots..(t-t_{m-1})}{k^m}\Delta^{0+m}u_{00}\right) \tag{4}$$

2.2. Formulation of the problem

We consider the following boundary value problem of one- dimensional heat equation.

$$u_t = u_{xx} \tag{5}$$

Subject to the following boundary conditions

$$u(0.t) = 0 \tag{6}$$

$$u(5,t) = 0 \tag{7}$$

$$u(x,0) = 25x^2 - x^4 \tag{8}$$

$where\quad 0 \le t \le 2.5\ and\ 0 \le x \le 5$

2.3. Solution of the problem
we take the interval of differencing of x as 1,
i.e., h=1.and the interval of differencing of t as $\frac{1}{2}$ $i.e\ k = \frac{1}{2}$
thus $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$ and
$t_0 = 0, t_1 = \frac{1}{2}, t_2 = 1, t_3 = \frac{3}{2}, t_4 = 2, t_5 = \frac{5}{2}$.
Drawing straight lines parallel to coordinate axis (t, x) we have 25 mesh points.

suppose
$u_{rs} = f(x_r, t_s)$ where $x_r = x_0 + rh$ and $t_s = t_0 + sk$. for (r, s = 0,1,2,3,4,5)
The boundary condition (2) gives
$$u_{00} = u_{01} = u_{02} = u_{03} = u_{04} = u_{05} = 0 \quad (9)$$
he boundary condition (3) gives
$$u_{50} = u_{51} = u_{52} = u_{53} = u_{54} = u_{55} = 0 \quad (10)$$
The initial condition (4) gives
$$u_{00} = 0, u_{10} = 24, u_{20} = 84, u_{30} = 144, u_{40} = 144, u_{50} = 0 \quad (11)$$
The other values are obtained from the recurrence relation
$u_{i,j+1} = u_{i,j} + \lambda(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$ for $\lambda = \frac{k}{h^2} = \frac{1}{2}$ which are shown in the Table 1.
Since the first and last column values of Table 1 are zero
The differences of different orders of $\Delta^{0+j}u_{0i}:(i, j = 0,1,2,3,4,5)$ are zero.
i.e. $\Delta^{0+j}u_{0i} = 0$. particularly
$$\Delta^{0+1}u_{00} = \Delta^{0+2}u_{00} = \Delta^{0+3}u_{00} = \Delta^{0+4}u_{00} = \Delta^{0+5}u_{00} = 0 \quad (12)$$

$$\Delta^{0+1}u_{50} = \Delta^{0+2}u_{50} = \Delta^{0+3}u_{50} = \Delta^{0+4}u_{50} = \Delta^{0+5}u_{50} = 0 \quad (13)$$
By considering the second column values of table 1 , the differences $\Delta^{0+j}u_{10}$ are obtained and shown in the table 2.
From Table 2, we have
$$\Delta^{0+1}u_{10} = 18, \Delta^{0+2}u_{10} = -18, \Delta^{0+3}u_{10} = 15, \Delta^{0+4}u_{10} = -18, \Delta^{0+5}u_{10} = 32.625 \quad (14)$$
By considering the third column values of Table 1, the differences of $\Delta^{0+j}u_{20}$, are obtained and shown in the table 3.
From Table 3, we have
$$\Delta^{0+1}u_{20} = 0, \Delta^{0+2}u_{20} = -6, \Delta^{0+3}u_{20} = -6, \Delta^{0+4}u_{20} = 29.25, \Delta^{0+5}u_{20} = -70.25 \quad (15)$$
Similarly by considering 4th and 5th columns we obtain
$$\Delta^{0+1}u_{30} = -30, \Delta^{0+2}u_{30} = -6, \Delta^{0+3}u_{30} = 31.5, \Delta^{0+4}u_{30} = -64.5, \Delta^{0+5}u_{30} = 117 \quad (16)$$
$$\Delta^{0+1}u_{40} = -72, \Delta^{0+2}u_{40} = 57, \Delta^{0+3}u_{40} = -60, \Delta^{0+4}u_{40} = 75.75, \Delta^{0+5}u_{40} = -108 \quad (17)$$
Repeating the above process for rows of table 1 , we obtain
$$\Delta^{1+0}u_{00} = 24, \Delta^{2+0}u_{00} = 36, \Delta^{3+0}u_{00} = -36, \Delta^{4+0}u_{00} = -24, \Delta^{5+0}u_{00} = 0 \quad (18)$$
$$\Delta^{1+0}u_{01} = 42, \Delta^{2+0}u_{01} = 0, \Delta^{3+0}u_{01} = -12, \Delta^{4+0}u_{01} = -48, \Delta^{5+0}u_{01} = 150 \quad (19)$$
$$\Delta^{1+0}u_{02} = 42, \Delta^{2+0}u_{02} = -6, \Delta^{3+0}u_{02} = -30, \Delta^{4+0}u_{02} = 45, \Delta^{5+0}u_{02} = -75 \quad (20)$$
$$\Delta^{1+0}u_{03} = 39, \Delta^{2+0}u_{03} = -18, \Delta^{3+0}u_{03} = 4.5, \Delta^{4+0}u_{03} = -27, \Delta^{5+0}u_{03} = 75 \quad (21)$$
$$\Delta^{1+0}u_{04} = 30, \Delta^{2+0}u_{04} = -6.75, \Delta^{3+0}u_{04} = -20.25, \Delta^{4+0}u_{04} = 35.25, \Delta^{5+0}u_{04} = -56.25 \quad (22)$$
$$\Delta^{1+0}u_{05} = 26.625, \Delta^{2+0}u_{05} = -13.5, \Delta^{3+0}u_{05} = 4.125, \Delta^{4+0}u_{05} = -17.25, \Delta^{5+0}u_{05} = 46.875 \quad (23)$$

The differences $\Delta^{m+n}u_{00}$ $(m, n = 1,2,3,4,5 \text{ and } m + n \leq 5)$
are obtained from the formula (3) are given by
$$\Delta^{1+1}u_{00} = 18, \Delta^{1+2}u_{00} = -18, \Delta^{2+1}u_{00} = -36, \Delta^{3+1}u_{00} = 24, \Delta^{1+3}u_{00} = 15,$$
$$\Delta^{2+2}u_{00} = 30, \Delta^{1+4}u_{00} = -18, \Delta^{4+1}u_{00} = -24, \Delta^{3+2}u_{00} = -42, \Delta^{2+3}u_{00} = 36 \quad (24)$$
The formula for interpolating polynomial in two variables (4) up to 5th difference (m=5) is
$$u(x,t) = u_{00} + \left[\frac{(x-x_0)}{h}\Delta^{1+0}u_{00} + \frac{(t-t_0)}{k}\Delta^{0+1}u_{00}\right]$$
$$+\frac{1}{2!}\left[\frac{(x-x_0)(x-x_1)}{h^2}\Delta^{2+0}u_{00} + \frac{2(x-x_0)(t-t_0)}{hk}\Delta^{1+1}u_{00} + \frac{(t-t_0)(t-t_1)}{k^2}\Delta^{0+2}u_{00}\right]$$

$$+\frac{1}{3!}\Big[\frac{(x-x_0)(x-x_1)(x-x_2)}{h^3}\Delta^{3+0}u_{00}+\frac{3(x-x_0)(x-x_1)(t-t_0)}{h^2k}\Delta^{2+1}u_{00}+\frac{3(x-x_0)(t-t_0)(t-t_1)}{hk^2}\Delta^{1+2}u_{00}+\frac{(t-t_0)(t-t_1)(t-t_2)}{k^3}\Delta^{0+3}u_{00}\Big]$$

$$+\frac{1}{4!}\Big[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{h^4}\Delta^{4+0}u_{00}+\frac{4(x-x_0)(x-x_1)(x-x_2)(t-t_0)}{h^3k}\Delta^{3+1}u_{00}+\frac{6(x-x_0)(x-x_1)(t-t_0)(t-t_1)}{h^2k^2}\Delta^{2+2}u_{00}+\frac{4(x-x_0)(t-t_0)(t-t_1)(t-t_2)}{hk^3}\Delta^{1+3}u_{00}+\frac{(t-t_0)(t-t_1)(t-t_2)(t-t_3)}{k^4}\Delta^{0+4}u_{00}\Big]$$

$$+\frac{1}{5!}\Big[\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{h^5}\Delta^{5+0}u_{00}+\frac{5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(t-t_0)}{h^4k}\Delta^{4+1}u_{00}+\frac{10(x-x_0)(x-x_1)(x-x_2)(t-t_0)(t-t_1)}{h^3k^2}\Delta^{3+2}u_{00}+\frac{10(x-x_0)(x-x_1)(t-t_0)(t-t_1)(t-t_2)}{h^2k^3}\Delta^{2+3}u_{00}+\frac{5(x-x_0)(t-t_0)(t-t_1)(t-t_2)(t-t_3)}{hk^4}\Delta^{1+4}u_{00}+\frac{(t-t_0)(t-t_1)(t-t_2)(t-t_3)(t-t_4)}{k^5}\Delta^{0+5}u_{00}\Big] \quad (25)$$

Substituting the values of $\Delta^{m+n}u_{00}$ from the equations (12), (18) and (24) in equation (25) and simplifying we obtain

$$u(x,t)=25x^2-x^4+36xt-36x(x-1)t-36xt\left(t-\frac{1}{2}\right)+8x(x-1)(x-2)t+30x(x-1)t\left(t-\frac{1}{2}\right)+20xt\left(t-\frac{1}{2}\right)(t-1)-2x(x-1)(x-2)(x-3)t-14x(x-1)(x-2)t\left(t-\frac{1}{2}\right)+24x(x-1)t\left(t-\frac{1}{2}\right)(t-1)-12xt\left(t-\frac{1}{2}\right)(t-1)\left(t-\frac{3}{2}\right)$$

. (26)

The required interpolating polynomial of $u(x,t)$ which is the approximate solution of heat equation (5) is given by equation (26).

III. TABLES

Table 1. Function Table.

t \ x	0	1	2	3	4	5
0	0	24	84	144	144	0
½	0	42	84	114	72	0
1	0	42	78	78	57	0
3/2	0	39	60	67.5	39	0
2	0	30	53.5	49.5	33.75	0
5/2	0	26.625	39.75	43.5	24.75	0

Table 2. Difference Table 1.

u_{1i}	$\Delta^{0+1}u_{1i}$	$\Delta^{0+2}u_{1i}$	$\Delta^{0+3}u_{1i}$	$\Delta^{0+4}u_{1i}$	$\Delta^{0+5}u_{1i}$
24					
→	18				
42	→	-18			
→	0	→	15		
42	→	-3	→	-18	
→	-3	→	-3	→	32.625
39	→	-6	→	14.625	
→	-9	→	11.625		
30	→	5.625			
→	-3.375				
26.625					

Table 3. Difference Table2.

u_{2i}	$\Delta^{0+1}u_{2i}$	$\Delta^{0+2}u_{2i}$	$\Delta^{0+3}u_{2i}$	$\Delta^{0+4}u_{2i}$	$\Delta^{0+5}u_{2i}$
84					
	→ 0				
84		→ -6			
	→ -6		→ -6		
78		→ -12		→ 29.25	
	→ -18		→ 23.25		→ -70.25
60		→ 11.25		→ -41.25	
	→ -6.75		→ -18		
53.25		→ -6.75			
	→ -13.5				
39.75					

IV. CONCLUSION

Problems considered in thermodynamics and engineering may yield parabolic equations. usually the analytical solution of such equations are in Fourier series. In this we obtain the solution as a polynomial in two variables.

REFERENCES

[1] Bender.C.M and Orszag.S.A., *Advanced mathematics methods for scientist and engineers* (McGraw-Hill. New York,1978).

[2] Collatz, L.,*The numerical treatment of differential equations* (Springer Verlag,Berlin,1996).

[3] Na,T.Y., *Computational methods in engineering boundary value problems* (Academic Press, New York,1979).

[4] Rama ChandraRao,P.S., Solution of a Class of Boundary Value Problems using Numerical Integration, *Indian journal of mathematics and mathematical sciences.vol.2.No2,2006,pp.137-146*.

[5] Rama Chandra Rao,P.S., Solution of fourth order of boundary value problems using spline functions, *Indian Journal of Mathematics and Mathematical Sciences.vol.2.No1,2006, pp.47-56*

[6] Ravikanth,A.S.V., *Numerical treatment of singular boundary value problems*, Ph.D. Thesis, National Institute of Technology, Warangal, India,2002.

[7] K.W. Mortan and D.F. Mayers. *Numerical solution of partial differential equations: An introduction* (Cambridge University press, Cambridge,England.1994).

[8] Jeffery Cooper, *Introduction to partial differential equations with matlab* (Birkhauser, Bosten, 1998).

[9] Clive A.J,Fletcher, *Computational techniques for fluid dynamics* (Springer-Verlag,Berlin,1998).

[10] J.B. Scorborough, *Numerical mathematical analysis* (Oxford and IBH Publishing Co. Pvt. Ltd.,1966)

[11] F.de Monte, Transient heat conduction in one-dimensional composite slab: A natural analytic approach, *International Journal of heat and mass transfer,Vol.43,2000,pp.3607-3619*.

[12] X Lu,P Tervola and M Viljanen, An efficient analytical solution to the transient heat conduction in one dimensional hollow composite cylinder, *J.Phy.A.Math.Gen.Vol.38,2005,pp.10145-10155*.