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Comparative Study of the Effect of Different Collocation Points on Legendre-Collocation Methods of solving Second-order Boundary Value Problems

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Abstract : *We seek to explore the effects of three basic types of Collocation points namely points at zeros of Legendre polynomials, equally-spaced points with boundary points inclusive and equally-spaced points with boundary point non-inclusive. Established in literature is the fact that type of collocation point influences to a large extent the results produced via collocation method (using orthogonal polynomials as basis function). We analyse the effect of these points on the accuracy of collocation method of solving second order BVP. For equally-spaced points we further consider the effect of including the boundary points as collocation points. Numerical results are presented to depict the effect of these points and the nature of problem that is best handled by each.*

Keywords – *Boundary points, Collocation method, Equally-spaced point, Legendre polynomial, Zeros of Legendre polynomial*

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I. INTRODUCTION

Classical orthogonal polynomials, established to be the most widely used orthogonal polynomials have over the years been used for solving scientific problems cutting across different fields of applications in Mathematics, physics and engineering. Of these polynomials, the most commonly used type – Jacobi polynomials (whose special cases includes; Chebyshev, Legendre and Gegenbauer polynomials) are the most extensively studied and widely applied sets [1]. This is basically due to a number of properties that these polynomials exhibit. For example the minimax property of the first-kind Chebyshev polynomials $T_n(x)$ makes it to stand out especially in the field of approximation [2].

This work however is to deploy the use of Legendre polynomial of the first kind in collocation methods of solving second order boundary value problems (BVP). The main purpose is to analyse the effect of different types of collocation points on the accuracy of this solution technique. The second order BVP considered is of the form:

$$\frac{d^2u}{dx^2} + q(x)\frac{du}{dx} + r(x)\,u = f(x) \qquad (1)$$

Subject to the boundary condition:

$$y(a) = A$$
$$y(b) = B \qquad (2)$$

q(x) r(x) and f(x) are real functions of independent variable *x*.

Equations (1) and (2) form a 2-point BVP of the second order and it finds relevant application in real life situation ranging from science to engineering fields where problems they model include; spring problems, electrical circuit problem, buoyancy problems, to mention a few. To these problems, arriving at a close-form solution are not always feasible for the mere fact that quite a good number of these real life problems do not have analytical solution and even in the availability of these solutions, it is well known that these are not amenable to direct numerical interpretation and hence limited in their usefulness in practical applications ([3], [4]). Also there are some of these differential equations for which the solution in terms of formula are so complicated that one often prefers to apply numerical methods ([5],[6],[7]). Owing to these facts, there is always the need to develop new numerical methods of solution and to improve on the existing ones.

A good number of research work had ever since been carried out on this problem, each providing numerical means of solution, for instance in [8], Legendre polynomial is applied via comparison technique while David [5] deployed Finite element method (FEM) in solving the same problem. In a bid to enhance results many authors had also applied collocation method with different means of improving its solutions (see [4],[6],[9]). In this work, we set to portray the comparative advantages of each of the collocation points and to display the nature of problem that each can handle best.

This paper is organised as follows: Section 2 represents brief introduction to Legendre polynomials and its basic properties, in section 3 we give a detailed account on collocation method and collocation points while implementation of the method is carried out in section 4. In section 5 illustrative examples are given and conclusion drawn in section 6.

II. LEGENDRE POLYNOMIALS

In this section a brief definition of Legendre polynomial is given, this is needed for easy reference and is necessary for the basic properties needed to establish our results. It is to be noted that the expression "Legendre polynomial" when no other qualification is attached exclusively refers to the Legendre polynomials $P_n(x)$ of the first kind. As established in literatures (see ref [8],[10]) Legendre polynomial are special cases of the Legendre function and they satisfy the Legendre equation defined as:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad n>0, \quad |x|<1 \qquad (3)$$

Solutions of this equation are called Legendre functions of order n. The general solution can be expressed as:

$$y = AP_n(x) + BQ_n(x) \qquad |x|<1 \qquad (4)$$

$P_n(x)$ and $Q_n(x)$ in (4) are respectively the Legendre functions of the first- and second-kind of order n.
The *nth* order polynomial $P_n(x)$ is generally given by the following equation:

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left(\frac{n!}{(2n)!}\right) \frac{(2n-2k)!}{k!\,(n-k)!(n-2k)!} x^{n-2k} \qquad (5)$$

Where n is the order of polynomial and $\lfloor \frac{n}{2} \rfloor$ signifies the integer part of $\frac{n}{2}$.

The basic property of Legendre polynomial is that these are orthogonal to each other with respect to weight function $w(x) = 1$ on [-1 1]. The first two polynomials are always the same in all cases but the higher orders are created with recursive formula:

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x), \qquad n = 1, 2, . . . \qquad (6)$$

With initial conditions:

$$P_0(x) = 1$$
$$P_1(x) = x$$

With the use of (6) and the associated conditions, first few Legendre polynomials are given as follows:

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

For problems that exist in intervals other than the natural interval, a shifted version of this polynomial for a general interval [*a b*] is obtained through the substitution:

$$x = \frac{2}{b-a}\left[t - \frac{b+a}{2}\right] \qquad a < b$$

$$t = \frac{(b-a)x}{2} - \frac{b+a}{2} \tag{7}$$

Which transform the interval [*a b*] of the *t*-axis into the interval [-1 1] of the *x*-axis, hence the new polynomial is likewise orthogonal. The recursive relation for the general interval is therefore given as:

$$P^*_{n+1}(x) = \frac{2n+1}{(n+1)}\left[\frac{(b-a)x+b+a}{2}\right]P^*_n(x) - \frac{n}{n+1}P^*_{n-1}(x), \qquad n = 1, 2, . . . \tag{8}$$

where $P^*_n(x)$ is a shifted Legendre polynomials valid within [*a*, b]

III. Legendre-Collocation Method

Collocation method as one of the broad class methods of weighted residual (MWR) evolved as a valuable technique for the solution of a broad class of problems. The technique as adapted in this paper involves constructing approximating trial solution of the form:

$$u_N(x;a) = \sum_{n=0}^{N} c_n P_n(x) \tag{9}$$

Where N is the degree of the trial solution, c_n are specialized coordinates called degree of freedom, $P_n(x)$ is Legendre polynomial of order *n*.

The technique in this method, demands that (9) is substituted into (1) to yield:

$$R(x;a) \Rightarrow \sum_{n=0}^{N} c_n P_n''(x) + q(x)\sum_{n=0}^{N} c_n P'(x) + r(x)P_n(x) - f(x) \neq 0 \tag{10}$$

The true meaning of equation (1) in contrasts to (10) is that if the exact solutions were substituted for $u(x)$, with all terms taken to the LHS, then the RHS would identically be zero over the entire domain. If other functions such as the approximate trial solution (9) were substituted for $u(x)$, the result would be non-zero function in equation (10) called the residual equation [5]. The method requires that for each undetermined parameter c_n , we choose a point x_i called the collocation point in the domain of the problem. As established in literatures, the point x_i can be located anywhere in the domain and on the boundary, not necessarily in any particular pattern but it might be reasonable to distribute them evenly [9]. Three methods of selecting these points are considered in the following sub-sections.

3.1 Collocation points at zeros of Legendre Polynomials

It is important to first consider the following ideas behind location and interlacing of zeros of Legendre polynomial and their associated properties. Plotting $P_n(x)$ for the first few values of n we observed the following:

a. All zeros of $P_n(x)$ lie in $-1 < x < 1$.
b. Between two consecutive zeros of $P_{n+1}(x)$ there is one of $P_n(x)$.
c. Between two consecutive zeros of $P_n(x)$ there is one of $P_{n+1}(x)$
d. Between the smallest zero of $P_n(x)$ and -1 there is one zero of $P_{n+1}(x)$ and between the largest zero of $P_n(x)$ and +1, there is one zero of $P_{n+1}(x)$.

According to the idea of Lanczos [11], collocating at the zeros of orthogonal polynomials requires that at the zeros of relevant orthogonal polynomial, the residual equation (10) is satisfied, thus yielding a number of collocation equations of the form:

$$\sum_{n=0}^{N} c_n P_n''(x_i) + q(x_i)\sum_{n=0}^{N} c_n P_n'(x_i) + r(x_i)P_n(x_i) - f(x_i) = 0 \qquad (11)$$

It is worthy of note that the polynomial $P_{N-1}(x)$ is used in obtaining the collocation points so as to yield N-1 zeros needed to collocate equation (10).

3.2 Collocating at equally-spaced point (Boundary points non-inclusive)

The technique here demands that instead of collocating at points on zeros of $P_{N-1}(x)$, the collocation points are determined by the use of:

$$x_i = a + \frac{(b-a)(i)}{N} \qquad i = 1, 2, \ldots, N-1 \qquad (12)$$

Where a and b are respectively the lower and upper bound of the interval, N is the chosen degree of the trial solution.

It is to be noted that equation (12) yields points that are located within the interval of consideration without the inclusion of boundary points a and b.

3.3 Collocating at equally-spaced point (Boundary points inclusive)

We hereby choose collocation points such that x_i is spread across the given interval with the boundary points included. These points are determined by the use of:

$$x_i = a + \frac{(b-a)(i)}{N-2} \qquad i = 0, 1, \ldots, N-2 \qquad (13)$$

where all parameters are as defined above.

IV. Implementation Of Legendre-Collocation Method

Collocating the residual equation (10) at any of the collocation points x_i discussed in preceding sections yields N-1 collocation equations of the form in equation (11). For the second order BVP in equations (1-2), the method of solution under discussion demands that two equations be fetched from boundary conditions (2), by imposing (2) on the trial solution (9). These equations from the BCs in conjunction N-1 equations in (11) yield a system of N+1 equations which are solved by the use of any of the algebraic system solvers to give numerical values of c_n, these are thereafter substituted back into trial solution (9) to yield the required approximate solution to problem (1-2).

V. Illustrative examples

Given below are numerical examples to illustrate the simplicity and the applicability of the discussed method.

Example 5.1

Solve the boundary value problem:

$$\frac{d^2u}{dx^2} - u = -4xe^x$$

$$u(0) = u(1) = 0$$

The analytical solution is: $u(x) = x(1-x)e^x$

Example 5.2
Consider the differential equation:

$$x^2 \frac{d^2u}{dx^2} - 2u = -x$$

Subject to boundary conditions:

$$u(2) = u(3) = 0$$

The analytical solution is : $u(x) = \frac{1}{38}\left(19x - 5x^2 - \frac{36}{x}\right)$

Example 5.3
Solve the second-order differential equation:

$$(1+x^2)\frac{d^2u}{dx^2} + 4x\frac{du}{dx} + 2u = 0$$

Subject to boundary conditions:

$$u(0) = 1$$
$$u(2) = 0.2$$

The exact solution is: $u(x) = \frac{1}{1+x^2}$

Table 5.1 Table of Error for Question 5.1

X	Collocation point	N=4	N=6	N=8	N=10
0	Points at zeros of $P_{N-1}(x)$	3.2960e-017	4.4385e-017	3.8291e-017	4.2486e-017
	Equally-spaced B.N	1.5613e-017	6.3290e-018	8.5338e-018	4.3749e-017
	Equally-spaced B.I	1.0408e-017	2.9057e-017	9.0939e-018	8.7264e-018
0.1	Points at zeros of $P_{N-1}(x)$	7.3418e-004	3.0938e-006	1.3390e-009	2.6482e-012
	Equally-spaced B.N	1.6076e-003	1.3046e-005	4.3653e-008	8.2341e-011
	Equally-spaced B.I	3.5346e-003	1.3956e-005	2.9512e-008	3.7309e-011
0.2	Points at zeros of $P_{N-1}(x)$	1.7550e-003	5.1467e-007	6.3551e-009	1.6953e-012
	Equally-spaced B.N	1.3052e-003	8.4257e-006	2.8280e-008	5.6958e-011
	Equally-spaced B.I	5.4162e-003	1.1680e-005	1.4000e-008	1.7597e-011
0.3	Points at zeros of $P_{N-1}(x)$	1.9616e-003	5.1000e-006	1.4458e-009	4.0379e-012
	Equally-spaced B.N	5.4312e-004	4.6808e-006	1.8213e-008	3.6240e-011
	Equally-spaced B.I	4.9633e-003	2.2892e-006	6.0356e-009	1.4176e-011
0.4	Points at zeros of $P_{N-1}(x)$	1.0944e-003	5.5075e-006	8.7763e-009	6.9652e-012
	Equally-spaced B.N	3.2694e-005	1.9304e-006	7.0636e-009	1.5265e-011
	Equally-spaced B.I	2.3783e-003	2.7181e-006	5.6798e-009	4.9766e-012
0.5	Points at zeros of $P_{N-1}(x)$	4.8826e-004	3.1752e-007	2.0787e-010	1.0020e-013
	Equally-spaced B.N	3.5775e-004	1.5408e-006	3.6262e-009	5.4176e-012
	Equally-spaced B.I	1.4758e-003	1.3212e-006	1.9832e-009	2.0260e-012
0.6	Points at zeros of $P_{N-1}(x)$	2.0708e-003	6.1160e-006	9.1262e-009	7.0867e-012
	Equally-spaced B.N	6.8840e-004	5.0220e-006	1.4358e-008	2.6151e-011
	Equally-spaced B.I	5.3361e-003	3.1520e-008	9.6272e-009	9.0723e-012
0.7	Points at zeros of $P_{N-1}(x)$	2.8833e-003	5.3757e-006	1.2064e-009	4.3765e-012
	Equally-spaced B.N	1.2986e-003	7.7828e-006	2.5623e-008	4.7297e-011
	Equally-spaced B.I	7.8428e-003	5.3638e-006	9.7931e-009	1.8364e-011
0.8	Points at zeros of $P_{N-1}(x)$	2.4545e-003	1.0109e-006	7.0619e-009	1.7165e-012
	Equally-spaced B.N	2.1272e-003	1.1677e-005	3.5808e-008	6.8249e-011
	Equally-spaced B.I	7.8917e-003	1.5515e-005	1.8202e-008	2.1656e-011
0.9	Points at zeros of $P_{N-1}(x)$	1.0206e-003	3.8559e-006	1.6525e-009	2.8945e-012
	Equally-spaced B.N	2.3689e-003	1.6551e-005	5.1897e-008	9.4368e-011
	Equally-spaced B.I	5.0440e-003	1.7445e-005	3.4758e-008	4.2550e-011
1.0	Points at zeros of $P_{N-1}(x)$	3.2960e-017	4.4385e-017	3.8291e-017	1.7956e-016
	Equally-spaced B.N	1.5613e-017	6.3290e-018	2.1351e-016	4.3749e-017
	Equally-spaced B.I	1.0408e-017	2.9057e-017	9.0939e-018	1.1975e-016

Table 5.2 Table of Error for Question 5.2

X	Collocation point	N=4	N=6	N=8	N=10
2.0	Points at zeros of $P_{N-1}(x)$	2.2204e-016	0	1.7764e-015	6.2172e-015
	Equally-spaced B.N	4.4409e-016	8.8818e-016	8.8818e-016	1.1546e-014
	Equally-spaced B.I	4.4409e-016	4.4409e-016	1.3323e-015	6.2172e-015
2.1	Points at zeros of $P_{N-1}(x)$	1.6734e-005	2.2511e-007	7.2800e-010	1.3195e-011
	Equally-spaced B.N	3.9775e-005	9.4586e-007	2.0701e-008	4.4617e-010
	Equally-spaced B.I	8.6139e-005	1.0232e-006	1.4230e-008	2.0697e-010
2.2	Points at zeros of $P_{N-1}(x)$	4.0414e-005	7.7577e-008	2.6683e-009	6.6994e-012
	Equally-spaced B.N	3.6666e-005	7.0486e-007	1.5185e-008	3.4225e-010
	Equally-spaced B.I	1.3508e-004	9.4624e-007	8.2339e-009	1.1685e-010
2.3	Points at zeros of $P_{N-1}(x)$	4.7938e-005	2.6433e-007	2.7736e-010	1.9521e-011
	Equally-spaced B.N	2.4057e-005	5.0963e-007	1.1736e-008	2.5920e-010
	Equally-spaced B.I	1.3635e-004	4.1336e-007	5.1518e-009	1.0368e-010
2.4	Points at zeros of $P_{N-1}(x)$	3.6033e-005	3.1440e-007	3.1654e-009	2.8535e-011
	Equally-spaced B.N	1.4636e-005	3.7013e-007	7.8748e-009	1.7443e-010
	Equally-spaced B.I	9.8246e-005	1.2457e-007	5.0505e-009	6.5191e-011
2.5	Points at zeros of $P_{N-1}(x)$	1.2346e-005	3.5056e-008	2.1961e-010	1.5886e-012
	Equally-spaced B.N	9.3080e-006	1.9475e-007	4.1657e-009	9.0822e-011
	Equally-spaced B.I	3.8908e-005	1.7684e-007	2.3706e-009	3.5344e-011
2.6	Points at zeros of $P_{N-1}(x)$	1.1421e-005	2.4762e-007	2.7978e-009	2.6564e-011
	Equally-spaced B.N	4.0567e-006	1.9395e-008	4.7603e-010	7.5346e-012
	Equally-spaced B.I	2.0513e-005	2.3331e-007	3.4290e-010	5.9213e-012
2.7	Points at zeros of $P_{N-1}(x)$	2.4839e-005	2.3697e-007	5.4209e-010	1.4194e-011
	Equally-spaced B.N	4.7243e-006	1.2217e-007	3.3357e-009	7.6248e-011
	Equally-spaced B.I	6.1205e-005	1.2841e-008	6.8530e-010	3.1661e-011
2.8	Points at zeros of $P_{N-1}(x)$	2.2947e-005	1.4063e-008	1.9072e-009	6.4622e-012
	Equally-spaced B.N	1.6039e-005	3.0551e-007	6.7756e-009	1.5818e-010
	Equally-spaced B.I	7.0887e-005	4.6112e-007	3.3492e-009	4.7768e-011
2.9	Points at zeros of $P_{N-1}(x)$	9.5744e-006	1.3148e-007	3.6659e-010	9.1793e-012
	Equally-spaced B.N	2.0967e-005	5.2456e-007	1.1687e-008	2.5401e-010
	Equally-spaced B.I	4.7125e-005	5.8867e-007	8.3021e-009	1.2041e-010
3.0	Points at zeros of $P_{N-1}(x)$	1.1102e-015	3.5527e-015	1.0658e-014	2.7534e-014
	Equally-spaced B.N	1.3323e-015	8.8818e-016	1.3323e-014	7.9936e-015
	Equally-spaced B.I	0	3.9968e-015	6.6613e-015	6.2172e-015

Table 5.3 Table of Error for Question 5.3

X	Collocation point	N=4	N=6	N=8	N=10
0	Points at zeros of $P_{N-1}(x)$	0	0	0	0
	Equally-spaced B.N	0	0	0	0
	Equally-spaced B.I	0	0	0	0
0.2	Points at zeros of $P_{N-1}(x)$	5.2505e-003	1.8755e-003	8.0674e-005	1.6641e-005
	Equally-spaced B.N	8.6995e-003	1.1856e-002	9.7631e-004	7.1869e-004
	Equally-spaced B.I	3.8744e-002	7.0400e-003	1.1262e-003	1.9757e-004
0.4	Points at zeros of $P_{N-1}(x)$	1.1488e-002	4.0070e-005	8.4109e-005	1.5204e-005
	Equally-spaced B.N	2.0488e-002	8.2752e-003	1.6290e-003	4.3615e-004
	Equally-spaced B.I	6.3717e-002	4.2743e-003	1.1358e-003	2.8918e-005
0.6	Points at zeros of $P_{N-1}(x)$	1.9646e-002	3.0789e-003	9.9106e-005	2.0685e-005
	Equally-spaced B.N	3.1296e-002	5.4423e-003	1.9550e-003	2.2220e-004
	Equally-spaced B.I	8.1109e-002	1.4565e-003	1.0614e-003	9.3062e-006
0.8	Points at zeros of $P_{N-1}(x)$	3.2668e-002	2.5644e-003	6.7116e-005	4.3130e-005
	Equally-spaced B.N	3.5301e-002	3.5831e-003	2.1238e-003	5.6518e-005
	Equally-spaced B.I	9.7684e-002	3.5117e-003	9.5064e-004	6.8536e-005
1.0	Points at zeros of $P_{N-1}(x)$	4.6667e-002	4.5375e-004	8.4893e-005	5.3482e-006
	Equally-spaced B.N	3.3762e-002	1.9507e-003	2.1399e-003	5.7759e-005
	Equally-spaced B.I	1.1194e-001	2.3302e-003	9.1538e-004	9.4774e-005
1.2	Points at zeros of $P_{N-1}(x)$	5.4188e-002	2.4344e-003	1.6187e-004	3.4291e-005
	Equally-spaced B.N	3.1759e-002	7.8444e-004	2.0831e-003	1.3265e-004
	Equally-spaced B.I	1.1737e-001	1.4991e-003	8.4862e-004	1.1009e-004
1.4	Points at zeros of $P_{N-1}(x)$	4.9710e-002	1.6867e-003	1.9942e-005	2.1672e-005
	Equally-spaced B.N	3.2693e-002	1.0407e-004	1.9979e-003	1.8018e-004
	Equally-spaced B.I	1.0796e-001	2.8758e-003	6.6958e-004	1.2531e-004
	Points at zeros of $P_{N-1}(x)$	3.3198e-002	3.5849e-004	1.3338e-004	5.5973e-006

1.6	Equally-spaced B.N	3.4733e-002	4.8514e-004	1.8577e-003	2.0683e-004
	Equally-spaced B.I	8.1759e-002	4.9399e-003	6.7189e-004	1.1176e-004
1.8	Points at zeros of $P_{N-1}(x)$	1.1721e-002	9.4708e-004	3.9418e-005	1.0252e-005
	Equally-spaced B.N	2.9195e-002	9.7570e-004	1.8788e-003	2.3217e-004
	Equally-spaced B.I	4.2464e-002	4.3731e-003	7.7360e-004	1.5467e-004
2.0	Points at zeros of $P_{N-1}(x)$	7.2164e-016	7.2164e-016	5.4956e-015	4.1633e-015
	Equally-spaced B.N	6.1062e-016	1.1657e-015	3.6082e-015	1.1047e-014
	Equally-spaced B.I	1.6653e-016	6.1062e-016	1.0936e-014	2.9421e-015

VI. CONCLUSION

In accordance with established facts, it is clearly observed from the results produced that these points are valid for collocation method via Legendre polynomial as trial functions. Errors accrued on each example are also minimal that these can be functionally applied in practical setting. A further study of these results likewise depicts that on a good number of problems, points at zeros of Legendre polynomial performed better, but at the boundaries region, equally-spaced points (with boundary points inclusive) produced better results. Also with a large degree of trial solution N, it is observed that with equally spaced point, better results are consistently achieved. We thus conclude that these techniques are not only effective numerical techniques but also efficient means of obtaining approximate solution that are close enough to the exact solution as to be useful in application, each with varying degree of accuracies. We therefore recommend that these techniques be extended to higher order problems.

References

[1]. Mason J.C. and Handscomb, D.C., *Chebyshev Polynomials,* Chapman & Hall – *CRC,* Roca Raton, London, New York, Washington D.C , 2003

[2]. Olagunju A. S., *Chebyshev Series Representation For Product of Chebyshev Polynomials and some notable functions,* IOSR Journal of Maths., vol. 2(2), pp 09-13, 2012

[3]. Richard Bronson, *Differential Equations*, Schaum's Outline series, McGRAW-HILL (2003).

[4]. Taiwo O. A, Olagunju A. S, Olotu O. T, Aro O. T, *Chebyshev Coefficient Comparison Method for the numerical solution of nonlinear BVPs*. Pioneer JAAM. 3(2), pp 101-110, 2011

[5]. David, S.B. Finite Element Analysis, from concepts to applications, *AT&T Bell Laboratory,* Whippany, New Jersey, (1987)

[6]. Taiwo O. A, Olagunju A. S, *Chebyshev methods for the numerical solution of fourth-order differential equations,* International Journal of Physical Sciences Vol. 7(13), pp. 2032 - 2037, 2012

[7]. Grewal, B.S. (2005): *Numerical Methods in Engineering and Science*, 7th ed. Kanna Publishers, Delhi.

[8]. Olagunju, A.S, Olaniregun D. G, *Legendre-coefficients Comparison methods for the Numerical solution of a class of Ordinary differential equation,* IOSR Journal of Maths. vol. 2(2), pp 14-19, 2012

[9]. Hermann, B. *Collocation Methods for Volterra integral and related functional Differential Equations,* Cambridge University press, New York, (2004)

[10]. Lanczos, C. *Legendre Versus Chebyshev polynomials.* Miller topics in Numerial analysis, Academic press, London, 1973.

[11]. Lanczos, C. *Applied Analysis,* Prentice Hall, Endlewood Cliffs, New Jersey, 1957.