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Some Summability Spaces of Double Sequences of Fuzzy Numbers

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Abstract: *In this article we introduce and study the notions of double Δ-lacunary strongly summable, double Δ- Cesàro strongly summable, double Δ- statistically convergent and double Δ-lacunary statistically convergent sequence of fuzzy numbers. Consequently we construct the spaces ${}_2L_\theta^F\left(\Delta\right), \sigma_2^F\left(\Delta\right)$, ${}_2S^F\left(\Delta\right)$ and ${}_2S_\theta^F\left(\Delta\right)$ respectively and investigate the relationship among these spaces. Further we show that ${}_2L_\theta^F\left(\Delta\right)$ and $\sigma_2^F\left(\Delta\right)$ are complete metric spaces.*

Keywords: *Sequence of fuzzy numbers; Difference sequence; lacunary strongly summable; Cesàro strongly summable; statistically convergent; lacunary statistically convergent; Completeness.*

I. Introduction

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [15] and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [7] where it was shown that every convergent sequence is bounded. Nanda [9] studied the spaces of bounded and convergent sequence of fuzzy numbers and showed that they are complete metric spaces. In [13] Savaş studied the space $m(\Delta)$, which we call the space of Δ-bounded sequence of fuzzy numbers and showed that this is a complete metric space.

Let D denote the set of all closed and bounded intervals $X = [a_1, b_1]$ on the real line R. For $X = [a_1, b_1] \in D$ and $Y = [a_2, b_2] \in D$, define $d(X, Y)$ by

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|).$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on R *i.e.* a mapping $X : R \to L(= [0,1])$ associating each real number t with its grade of membership $X(t)$.

The α- level set $[X]^\alpha$ set of a fuzzy real number X for $0 < \alpha \leq 1$, defined as

$X^\alpha = \{ t \in R : X(t) \geq \alpha\}$.

A fuzzy real number X is called *convex*, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper semi- continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon))$, for all $a \in L$ is open in the usual topology of R.

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $L(R)$.

The absolute value $|X|$ of $X \in L(R)$ is defined as (see for instance Kaleva and Seikkala [2])

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t > 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Let $\bar{d} : L(R) \times L(R) \to R$ be defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^\alpha, Y^\alpha).$$

Then $\bar{d}$ defines a metric on $L(R)$.

For $X, Y \in L(R)$ define

$X \leq Y$ iff $X^\alpha \leq Y^\alpha$ for any $\alpha \in [0, 1]$.

A subset E of $L(R)$ is said to be bounded above if there exists a fuzzy number M, called an upper bound of E, such that $X \leq M$ for every $X \in E$. M is called the least upper bound or supremum of E if M is an upper bound and M is the smallest of all upper bounds. A lower bound and the greatest lower bound or infimum are defined similarly. E is said to be bounded if it is both bounded above and bounded below.

II. Definitions and Background

A sequence $X = (X_k)$ of fuzzy numbers is a function X from the set N of all positive integers into $L(R)$. The fuzzy number X_k denotes the value of the function at $k \in N$ and is called the k-th term or general term of the sequence.

A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\bar{d}\,(X_k, X_0) < \varepsilon \quad \text{for } k > n_0$$

The set of convergent sequences is denoted by c^F. $X = (X_k)$ of fuzzy numbers is said to be a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\bar{d}\,(X_k, X_l) < \varepsilon \quad \text{for } k, l > n_0$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be bounded if the set $\{X_k: k \in N\}$ of fuzzy numbers is bounded and the set of bounded sequences is denoted by ℓ_∞^F.

Functional analytic studies of the space of strongly Cesàro summable sequences of complex terms and other closely related spaces of strongly summable sequences can be found in [5]

The notion of difference sequence of complex terms was introduced by Kizmaz [6]. Tripathy and Esi [16], Tripathy, Esi and Tripathy [17] and many others.

The idea of the statistical convergence of sequence was introduced by Fast [3] and Schoenberg [12] independently in order to extend the notion of convergence of sequences. It is also found in Zygmund [19]. Later on it was linked with summability by Fridy and Orhan [4], Maddox [8], Rath and Tripathy [11] and many others. In [10] Nuray and Savaş extended the idea to sequences of fuzzy numbers and discussed the concept of statistically Cauchy sequences of fuzzy numbers. In this article we extend these notions to difference sequences of fuzzy numbers.

The natural density of a set K of positive integers is denoted by $\delta(K)$ and defined by

$$\delta(K) = \lim_n \frac{1}{n} \operatorname{card}\{k \le n : k \in K\}$$

A sequence $X = (X_k)$ of fuzzy numbers is said to be statistically convergent to a fuzzy number X_0 if for every $\varepsilon > 0$, $\lim_n \frac{1}{n} \operatorname{card}\{k \le n : d(X_k, X_0) \ge \varepsilon\} = 0$. We write st-lim $X_k = X_0$.

A fuzzy double sequence is a double infinite array of fuzzy numbers. we denote a fuzzy double sequence by $(X_{k,l})$, where $X_{k,l}'s$ are fuzzy numbers for each $k, l \in N$. Throughout the article 2^{W^F} denote the set of all double sequences of fuzzy numbers.

A double sequence $(X_{k,l})$ of fuzzy numbers is said to be double Δ-convergent in Pringsheim sence to a fuzzy real number X_0 if for each $\varepsilon > 0$ there exist $k_0, l_0 \in N$ such that
$\bar{d}(\Delta X_{k,l}, X_0) > \varepsilon \quad for\ all\ k \ge k_0 , l \ge l_0$. We write $P - lim\Delta X_{k,l} = X_0$.

Where $\quad \Delta X_{k,l} = X_{k,l} - X_{k,l+1} - X_{k+1,l} + X_{k+1,l+1}$

A double sequence $(X_{k,l})$ of fuzzy numbers is said to be double Δ- null in Pringsheim sence if $P - lim\Delta X_{k,l} = \bar{0}$.

A double sequence $(X_{k,l})$ of fuzzy numbers is said to be double Δ- bounded in Pringsheim sence if $\quad sup_{k,l}\bar{d}(\Delta X_{k,l}, X_0) < \infty$.

A double sequence $(X_{k,l})$ of fuzzy numbers is said to be double Δ- statistically convergent to X_0 if for each $\varepsilon > 0$,

$$P - \lim_{m,n} \frac{1}{mn} card\{(k,l):\ k \le m , l \le n\ and\ \bar{d}(\Delta X_{k,l} , X_0) \ge \varepsilon\} = 0$$

We write $st_2 -\ lim\Delta X_{k,l} = X_0$

A double sequence $\theta_{r,s} = (\alpha_r, \beta_s\ )$ is said to be double lacunary if there exists sequences $(\alpha_r)\ and\ (\beta_s)$ of non negative integers such that

$$v_r = \alpha_r - \alpha_{r-1} \to \infty\ \ as\ \ r \to \infty\ ,\ \ \alpha_0 = 0$$
$$v_s = \beta_s - \beta_{s-1} \to \infty\ \ as\ \ s \to \infty\ ,\ \ \beta_0 = 0$$

Let $v_{r,s} = v_r v_s$, $\theta_{r,s}$ is obtain by $I_{r,s} = \{ (x,y) : \alpha_{r-1} < x \le \alpha_r \text{ and } \beta_{s-1} < y \le \beta_s \}$
Then for double lacunary sequence $\theta_{r,s}$, we define :

$$ {}_2L_\theta^F(\Delta) = \left\{ X = \left(X_{k,l}\right) \in 2^{W^F} : P - \lim_{r,s} \frac{1}{v_{r,s}} \sum_{k,l \in I_{r,s}} \bar{d}\left(\Delta X_{k,l}, X_0\right) = 0 \text{ for some } X_0 \right\} $$

If $X \in {}_2L_\theta^F(\Delta)$, then we say that X is double Δ- lacunary strongly summable double sequence of fuzzy numbers .

A double sequence $X = (X_{k,l}) \in 2^{W^F}$ is said to be double Δ- Cesaro strongly summable if $X \in \sigma_2^F(\Delta)$,where

$$ \sigma_2^F(\Delta) = \left\{ X = \left(X_{k,l}\right) \in 2^{W^F} : P - \lim_{m,n \to \infty,\infty} \frac{1}{mn} \sum_{k,l=1,1}^{m,n} \bar{d}\left(\Delta X_{k,l}, X_0\right) = 0 \text{ for some } X_0 \right\} $$

A double sequence $X = (X_{k,l}) \in 2^{W^F}$ is said to be double Δ- statistically convergent if $X \in {}_2S^F(\Delta)$, where

$$ {}_2S^F(\Delta) = \left\{ X = \left(X_{k,l}\right) \in 2^{W^F} : P - \lim_{m,n} \frac{1}{mn} \operatorname{card}\left\{(k,l) : k \le m, l \le n \text{ and } \bar{d}\left(\Delta X_{k,l}, X_0\right) \ge \varepsilon\right\} = 0 \text{ for each } \varepsilon \right. $$

A double sequence $X = (X_{k,l}) \in 2^{W^F}$ is said to be double Δ- lacunary statistically convergent if $X \in {}_2S_\theta^F(\Delta)$ where,

$$ {}_2S_\theta^F(\Delta) = \left\{ X = \left(X_{k,l}\right) \in 2^{W^F} : P - \lim_{r,s} \frac{1}{v_{r,s}} \operatorname{card}\left\{(k,l) \in I_{r,s} : \bar{d}\left(\Delta X_{k,l}, X_0\right) \ge \varepsilon\right\} = 0 \text{ for each } \varepsilon > 0 \right\} $$

III. Main results :

Theorem 3.1: Let $\theta_{r,s}$ be a double lacunary sequence . Then if a double sequence $X = (X_{k,l})$ is double Δ- lacunary strongly summable then it is double Δ- lacunary statistically convergent .

Proof : Suppose $X = (X_{k,l})$ is double Δ- lacunary strongly summable to X_0 . Then ,

$$ P - \lim_{r,s} \frac{1}{v_{r,s}} \sum_{k,l \in I_{r,s}} \bar{d}\left(\Delta X_{k,l}, X_0\right) = 0 \text{ for some } X_0 $$

Now the result follows from the following inequality :

$$ \sum_{k,l \in I_{r,s}} \bar{d}\left(\Delta X_{k,l}, X_0\right) \ge \varepsilon \operatorname{card}\left\{(k,l) \in I_{r,s} : \bar{d}\left(\Delta X_{k,l}, X_0\right) \ge \varepsilon\right\} $$

Theorem 3.2: If a double sequence $X = (X_{k,l})$ is double Δ- bounded and double Δ- statistically convergent, then it is double Δ- Cesaro strongly summable .

Proof : Suppose $X = (X_{k,l})$ is double Δ- bounded and double Δ- statistically convergent to X_0 . Since $X = (X_{k,l})$ is double Δ- bounded, we can find a fuzzy number M such that

$$ \bar{d}\left(\Delta X_{k,l}, X_0\right) \le M \quad \text{for all } k, l \in N $$

Again $X = (X_{k,l})$ is double Δ- statistically convergent to X_0, for each $\varepsilon > 0$ such that

$$ \text{P-}\lim_{m,n} \frac{1}{mn} \operatorname{card}\left\{(k,l) : k \le m, l \le n \text{ and } \bar{d}\left(\Delta X_{k,l}, X_0\right) \ge \varepsilon\right\} = 0 $$

Now the result follows from the following inequality :

$$\frac{1}{mn}\sum_{\substack{1\le k\le m\\1\le l\le n}}\bar{d}\left(\Delta X_{k,l}\,,X_0\right)=\frac{1}{mn}\sum_{\substack{1\le k\le m\\1\le l\le n\\\bar{d}(\Delta X_{k,l}\,,X_0)\ge\varepsilon}}\bar{d}\left(\Delta X_{k,l}\,,X_0\right)+\frac{1}{mn}\sum_{\substack{1\le k\le m\\1\le l\le n\\\bar{d}(\Delta X_{k,l}\,,X_0)<\varepsilon}}\bar{d}\left(\Delta X_{k,l}\,,X_0\right)$$

$$\le\frac{M}{mn}card\left\{(k,l):\ k\le m\,,l\le n\ and\ \bar{d}\left(\Delta X_{k,l}\,,X_0\right)\ge\varepsilon\right\}+\varepsilon$$

This completes the proof.

Theorem 3.3: Let $\theta_{r,s}$ be a double lacunary sequence . Then if a double sequence $X=(X_{k,l})$ is double Δ-bounded and double Δ- lacunary statistically convergent, then it is double Δ- lacunary strongly summable.

Proof : Proof follows by similar arguments as applied to prove above theorem .

Theorem 3.4: Let $\theta_{r,s}$ be a double lacunary sequence and $X=(X_{k,l})$ is double Δ- bounded . Then $X=(X_{k,l})$ is double Δ- lacunary statistically convergent if and only if it is double Δ- lacunary strongly summable .

Proof: Proof follows by combining Theorem3.1 and Theorem3.3 .

Theorem 3.5: ${}_2L_\theta^F\left(\Delta\right)$ is **a** complete metric space under the metric g defined by –

$$g(X,Y)=sup_{r,s}\left[\frac{1}{v_{r,s}}\sum_{k,l\in I_{r,s}}\bar{d}(\Delta X_{k,l},\Delta Y_{k,l})\right]$$

Proof: It is easy to see that $\bar{d}$ is a metric on ${}_2L_\theta^F\left(\Delta\right)$. To prove completeness. Let (X^i) be a Cauchy sequence in ${}_2L_\theta^F\left(\Delta\right)$, where $X^i=\left(X_{k,l}^i\right)$ for each $i\in N$. Therefore for each ε > 0 . there exist a positive integer n_0 such that

$$g(X^i,Y^j)=sup_{r,s}\left[\frac{1}{v_{r,s}}\sum_{k,l\in I_{r,s}}\bar{d}(\Delta X_{k,l}^i,\Delta X_{k,l}^j)\right]<\varepsilon\ \ for\ all\ \ i,j\ge\ n_0$$

It follows that ,

$$\frac{1}{v_{r,s}}\sum_{k,l\in I_{r,s}}\bar{d}(\Delta X_{k,l}^i,\Delta X_{k,l}^j)<\varepsilon\ \ \ for\ all\ \ i,j\ge\ n_0,for\ all\ \ r,s\in N$$

Hence

$$\bar{d}\left(\Delta X_{k,l}^i,\Delta X_{k,l}^j\right)<\varepsilon\ \ for\ all\ \ k,l\in N$$

This implies that $(\Delta X_{k,l}^i)$ is a Cauchy sequence in L(R) . But L(R) is complete and so $(\Delta X_{k,l}^i)$ is convergent in L(R).

For simplicity, let

$$\lim_{i\to\infty}\Delta X_{k,l}^i=\ N_{k,l}\ \ for\ \ k,l\ge 1$$

Considering $k,l=1,2,3\ldots r,s\ldots$, we can easily conclude that

$$\lim_{i\to\infty}X_{k,l}^i=\ X_{k,l}\ \ ,\ \ exists\ \ for\ all\ k,l\ge 1$$

It remains to show $X=(X_{k,l})\in\ {}_2L_\theta^F\left(\Delta\right)$

Now one can find that,

$$\lim_{j\to\infty}\frac{1}{v_{r,s}}\sum_{k,l\in I_{r,s}}\bar{d}(\Delta X_{k,l}^i,\Delta X_{k,l}^j)<\varepsilon\ \ for\ all\ \ \ i\ge\ n_0,for\ all\ \ r,s\in N$$

Thus,

$$\lim_{j\to\infty}\frac{1}{v_{r,s}}\sum_{k,l\in I_{r,s}}\bar{d}(\Delta X^{i}_{k,l},\Delta X_{k,l}) < \varepsilon \ for\ all\ \ i \geq\ n_0, for\ all\ \ r,s\in N$$

This implies that

$g(X^i, X) < \varepsilon$ for all $i \geq\ n_0$

This shows that, $X = (X_{k,l}) \in\ {}_2L^F_\theta\left(\Delta\right)$

This completes the proof. .

Theorem 3.6: $\sigma^F_2\left(\Delta\right)$ is **a** complete metric space under the metric h defined by –

$$h(X,Y) = sup_{r,s}\left[\frac{1}{rs}\sum_{\substack{1\leq k\leq r\\1\leq l\leq s}}\bar{d}(\Delta X_{k,l},\Delta Y_{k,l})\right]$$

Proof: Proof is same with the above Theorem. In fact $\theta_{r,s} = (2^r, 2^s)\,; r,s = 1,2,3\ ....$,

${}_2L^F_\theta\left(\Delta\right) = \sigma^F_2\left(\Delta\right)$

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