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# The Value Distribution of Some Differential Polynomials

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***Abstract:*** *We prove a value distribution theorem for meromorphic functions having few poles from which we obtain several interesting results which improve some results of W. Doeringer, C.C.Yang, A.P.Singh, G.P.Barker and others.*

## I. Introduction And Main Results

Let f be a transcendental meromorphic function in the plane.

A **monomial** in f, is an expression of the form $M[f]=f^{n_0}\left(f^{(1)}\right)^{n_1}\ldots\left(f^{(k)}\right)^{n_k}$ where $n_0$, $n_1$, $n_2,\ldots,n_k$ are non negative integers.

$\gamma_M = n_0+n_1+n_2+\ldots+n_k$ is called the degree of the monomial and $\Gamma_M = n_0+2n_1+\ldots+(k+1)n_k$, the weight.

If $M_1[f], M_2[f],\ldots, M_n[f]$ denote monomials in f, then,

Q[f] = $a_1M_1[f]+a_2M_2[f]+\ldots+a_nM_n[f]$, where $a_i\neq 0 (i=1,2,\ldots,n)$, is called a **differential polynomial in f** of degree $\gamma_Q = \mathrm{Max}\{\gamma_{M_j} : 1\le j\le n\}$ and weight $\Gamma_Q = Max\{\Gamma_{M_j} : 1\le j\le n\}$

Also, we call the numbers $\underline{\gamma_Q} = \min_{1\le j\le n}\gamma_{M_j}$ and k (the order of the highest derivative of f) the **lower degree** and the order of Q[f] respectively. If $\underline{\gamma_Q}=\gamma_Q$, $Q[f]$ is called a **homogeneous differential**

## II. polynomial.

W.K.Hayman in his well known problem book 'Problems in Function Theory' has raised some interesting open problems related to the value distribution of differential polynomials.

In 1988, Hong-XunYi[6] proved the following result:

**Theorem A:** Let f be transcendental meromorphic function in the plane and $Q_1[f]\not\equiv 0$, $Q_2[f]\not\equiv 0$ be differential polynomials in f.

Let $P_1[f]=a_nf^n+a_{n-1}f^{n-1}+\ldots\ldots\ldots+a_0 \quad (a_n(z)\not\equiv 0)$

If $F=P_1[f]Q_1[f]+Q_2[f]$, then

$$\left(n-\gamma_{Q_2}\right)T(r,f)\le \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{P_1[f]}\right)+\left(\Gamma_{Q_2}-\gamma_{Q_2}+1\right)\overline{N}(r,f)+S(r,f).$$

Thinking on the same lines, we considered a different combination of differential polynomials and obtained an interesting result which generalizes the result of Hong-Xun Yi.

The following theorem is our main result.

**Theorem 1:** Let f be transcendental meromorphic function in the plane and $Q_1[f]\not\equiv 0$, $Q_2[f]\not\equiv 0$ be differential polynomials in f.

Let $P_1[f]=a_nf^n+a_{n-1}f^{n-1}+\ldots\ldots\ldots+a_0 \quad (a_n(z)\not\equiv 0)$ (1)

and $P_2[f]=b_mf^m+b_{m-1}f^{m-1}+\ldots\ldots\ldots+b_0 \quad (b_m(z)\not\equiv 0)$ (2)

where n > m.

If $F=P_1[f]Q_1[f]+P_2[f]Q_2[f]$ (3)

Then, $\left(n-\gamma_{Q_2}\right)T(r,f) \le \overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{P_1[f]}\right)+3\overline{N}\left(r,\frac{1}{P_2[f]}\right)+\left(\Gamma_{Q_2}-\gamma_{Q_2}+1\right)\overline{N}(r,f)+S(r,f).$

To prove the above Theorem, we require the following Lemmas.

**Lemma 1[2]:** Suppose $P_1[f]$ is as in (1).
Then, $m(r, P_1[f]) = n\, m(r, f) + S(r, f)$

**Lemma 2 [4]:** Let f(z) be a transcendental meromorphic function, P[f] and Q[f] be differential polynomials in f. If degree of Q[f] is at most n and $f^n P[f]=Q[f]$, then , $m(r, P[f]) = S(r, f)$.

**Lemma 3 [3]:** If P[f] is a homogeneous differential polynomial of degree n, then,

$$m\left(r,\frac{P[f]}{f^n}\right)=S(r,f).$$

**Lemma 4 [6]:** Suppose that Q[f] is a differential polynomial in f. Let $z_0$ be a pole of f of order m and not a zero or a pole of the co-efficients of Q[f]. Then $z_0$ is a pole of Q[f] of order atmost $m\gamma_Q+\left(\Gamma_Q-\gamma_Q\right)$.

**Lemma 5 [11]:** If Q[f] is a differential polynomial in f with arbitrary meromorphic co-efficients $q_j$, then

$$m(r,Q[f]) \le \gamma_Q m(r,f)+\sum_{j=1}^{n} m(r,q_j)+S(r,f).$$

**Proof of Theorem 1:**

We have $F=P_1[f]\,Q_1[f]+P_2[f]\,Q_2[f]$

Therefore, $F'=\frac{F'}{F}P_1[f]\,Q_1[f]+\frac{F'}{F}P_2[f]\,Q_2[f]$

Also, $F'=P_1[f]\left(Q_1[f]\right)'+Q_1[f]\left(P_1[f]\right)'+P_2[f]\left(Q_2[f]\right)'+Q_2[f]\left(P_2[f]\right)'$

Hence, we have

$$\frac{F'}{F}P_1[f]\,Q_1[f]+\frac{F'}{F}P_2[f]\,Q_2[f]=P_1[f]\left(Q_1[f]\right)'+Q_1[f]\left(P_1[f]\right)'$$
$$+P_2[f]\left(Q_2[f]\right)'+Q_2[f]\left(P_2[f]\right)'$$

Therefore,

$$P_1[f]\left[\frac{F'}{F}Q_1[f]-\left(Q_1[f]\right)'-\frac{\left(P_1[f]\right)'Q_1[f]}{P_1[f]}\right]=P_2[f]\left[\left(Q_2[f]\right)'-\frac{F'}{F}Q_2[f]+\frac{\left(P_2[f]\right)'Q_2[f]}{P_2[f]}\right].$$

Or, $$P_1[f].\left\{\frac{1}{P_2[f]}\left[\frac{F'}{F}Q_1[f]-\left(Q_1[f]\right)'-\frac{\left(P_1[f]\right)'Q_1[f]}{P_1[f]}\right]\right\}=\left[\left(Q_2[f]\right)'-\frac{F'}{F}Q_2[f]+\frac{\left(P_2[f]\right)'Q_2[f]}{P_2[f]}\right],$$

which is of the form, $P_1[f]\,Q*[f]=Q[f]$, (4)

where, $$Q^*[f]=\frac{F'}{FP_2[f]}Q_1[f]-\frac{\left(Q_1[f]\right)'}{P_2[f]}-\frac{\left(P_1[f]\right)'Q_1[f]}{P_1[f]P_2[f]}$$

and $$Q[f]=\left(Q_2[f]\right)'-\frac{F'}{F}Q_2[f]+\frac{\left(P_2[f]\right)'Q_2[f]}{P_2[f]}. \qquad (5)$$

Without loss of generality, let us assume that $Q[f]\not\equiv 0.$

By Lemma 2, $m(r,Q*[f])=S(r,f)$

Again from (4), $P_1[f]=\frac{Q[f]}{Q*[f]}$

Therefore, $m(r,P_1[f]) \le m(r,Q[f]) + m\left(r,\frac{1}{Q^*[f]}\right)$ (6)

Again by Lemma 5 and(5), $m(r,Q[f]) \le \gamma_{Q_2} m(r,f) + S(r,f)$ (7)

From the First Fundamental Theorem, we have

$$m\left(r,\frac{1}{Q^*[f]}\right) = N(r,Q^*[f]) - N\left(r,\frac{1}{Q^*[f]}\right) + S(r,f) \quad (8)$$

Also, all the poles of $Q^*[f]$ occur at the zeros of F, $P_1[f]$ and $P_2[f]$, the pole of f, and the zeros and poles of the co-efficients. Suppose $z_0$ is a pole of f of order m. Then $z_0$ is a pole $P_1[f]$ of order mn.

From Lemma 4, $z_0$ is a pole of Q[f] of order atmost $m\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)$

If $z_0$ is a pole of Q*[f], since $Q^*[f] = \frac{Q[f]}{P_1[f]}$,

$z_0$ is a pole of Q*[f] of order atmost $m\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1) - mn$

$$= (\Gamma_{Q_2} - \gamma_{Q_2} + 1) - m(n - \gamma_{Q_2})$$

If $z_0$ is a pole of Q*[f] then since $\frac{1}{Q^*[f]} = \frac{P_1[f]}{Q[f]}$,

$z_0$ is a zero of $\frac{1}{Q^*[f]}$ of order atleast $mn - \{m\gamma_{Q_2} + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\}$

$$= m(n - \gamma_{Q_2}) - (\Gamma_{Q_2} - \gamma_{Q_2} + 1)$$

Thus, we have,

$$N(r,Q^*[f]) - N\left(r,\frac{1}{Q^*[f]}\right) \le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{P_1[f]}\right) + 3\overline{N}\left(r,\frac{1}{P_2[f]}\right)$$
$$+ (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) - (n - \gamma_{Q_2})N(r,f) + S(r,f) \quad (9)$$

In view of (7), (8), (9), the equation (6) becomes ,

$$nm(r,f) \le \gamma_{Q_2} m(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{P_1[f]}\right) + 3\overline{N}\left(r,\frac{1}{P_2[f]}\right)$$
$$+ (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) - (n - \gamma_{Q_2})N(r,f) + S(r,f).$$

Hence, we have

$$(n - \gamma_{Q_2})T(r,f) \le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{P_1[f]}\right) + 3\overline{N}\left(r,\frac{1}{P_2[f]}\right)$$
$$+ (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) + S(r,f).$$

Hence the result.

Putting $P_1[f] = f^n$ and $P_2[f] = f^{n-1}$ in the above theorem, we get the following.

**Theorem 2:** Let f, $Q_1[f], Q_2[f]$ be as defined in Theorem 1.

Let $F = f^n Q_1[f] + f^{n-1} Q_2[f]$.

Then,

$$(n - \gamma_{Q_2})T(r,f) \le \overline{N}\left(r,\frac{1}{F}\right) + 4\overline{N}\left(r,\frac{1}{f}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\overline{N}(r,f) + S(r,f)$$

If $P_2[f] \equiv 1$, we get the following.

**Theorem 3:** Let $P_1[f], Q_1[f]$ and $Q_2[f]$ be as defined in Theorem 1.

If $F = P_1[f]Q_1[f] + Q_2[f]$, then

$$\left(n-\gamma_{Q_2}\right)T(r,f) \le \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{P_1[f]}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)\overline{N}(r,f) + S(r,f)$$

which is the result of Hong Xun Yi [5].
As an application of theorem 2, we have the following.

**Theorem 4:** Let $F = f^n Q[f]$ where $Q[f]$ is a differential polynomials in f.
If n ≥ 1, then $\rho_F = \rho_f$ and $\lambda_F = \lambda_f$

**Proof :** If $F = f^n Q[f]$ ,then by theorem2, we have

$$n\,T(r,f) \le \overline{N}(r,\frac{1}{F}) + 4\overline{N}(r,\frac{1}{f}) + \overline{N}(r,f) + S(r,f).$$

Clearly, the zeros and poles of f are that of F respectively.

Therefore, $4\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,f) \le 4\overline{N}\left(r,\frac{1}{F}\right) + \overline{N}(r,F) + S(r,f)$

Therefore, $n\,T(r,f) \le \overline{N}(r,\frac{1}{F}) + 4\overline{N}(r,\frac{1}{f}) + \overline{N}(r,f) + S(r,f)$

$$\le 6\;T(r,F) + S(r,f)$$

Therefore, $T(r,f) = O\{T(r,F)\}$ as $r \to \infty$.
Also, we know that $T(r,F) = O\{T(r,f)\}$ as $r \to \infty$.
Hence the Theorem.

**Theorem 5:** No transcendental meromorphic function f with $\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right)$=S(r,f)

can satisfy an equation of the form

$$F= a_1 f^n Q_1[f] + a_2 f^{n-1} Q_2[f] + a_3 = 0 \qquad (9)$$

where $a_1 \not\equiv 0$, $a_3 \not\equiv 0$, n is a positive integer with $n > \gamma_{Q_2}$, $Q_1[f]$ and $Q_2[f]$ are differential polynomials in f.

**Proof:** Suppose there exists a transcendental meromorphic function f satisfying (9).
Then by theorem 2, we have,

$$\left(n-\gamma_{Q_2}\right)T(r,f) \le \overline{N}\left(r,\frac{1}{F}\right) + 4\overline{N}\left(r,\frac{1}{f}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)\overline{N}(r,f) + S(r,f)$$

$$\le 5\,\overline{N}(r,\frac{1}{f}) + S(r,f)$$

Or, $n - \gamma_{Q_2} \le \dfrac{5\,\overline{N}(r,\frac{1}{f}) + S(r,f)}{T(r,f)} = \dfrac{S(r,f)}{T(r,f)}$ by hypothesis.

Or, $n - \gamma_{Q_2} \le 0$, which implies $n \le \gamma_{Q_2}$ which contradicts the choice of n.
Hence the result.

This improves our earlier result namely

**Theorem A [10]:** No transcendental meromorphic function f with N(r, f) = S(r, f) can satisfy the equation

$$a_1(z)[f(z)]^n \pi_1(f) + a_2(z)[f(z)]^{n-1}\pi_2(f) + a_3(z) = 0,$$

where $\pi_1(f)$ and $\pi_2(f)$ are differential polynomials of degree n and n – 1 respectively and n > 1, and $Max\{\deg(f^{n-1}\pi_2(f))\} < n.$ and $a_1(z) \not\equiv 0.$

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