*IOSR Journal of Mathematics (IOSR-JM)*
*e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 9, Issue 5 (Jan. 2014), PP 25-29*
*www.iosrjournals.org*

# Simple Partially Ordered Ternary Semigroups

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***Abstract:*** *In this paper the terms, simple po ternary semigroup, globally idempotent po ternary semigroup, semisimple element, regular element, left regular element, lateral regular element, right regular element, completely regular element, intra regular element in a poternary semigroup are introduced. It is proved that, if a is a completely regular element of a po ternary semigroup T then a is left regular, lateral regular and right regular. It is proved that in any po ternary semigroup T, the following are equivalent (1) Principal po ideals of T form a chain. (2) Po ideals of T form a chain. It is proved that a po ternary semigroup T is simple po ternary semigroup if and only if ( TTaTT ] = T for all $a \in T$.*

***Mathematical subject classification (2010) :*** *20M07; 20M11; 20M12.*
***Key Words:*** *regular element, completely regular element, intra regular element in a po ternary semigroup, simple po ternary semigroup, globally idempotent po ternary semigroup, semisimple element.*

## I. Introduction :

The algebraic theory of semigroups was widely studied by CLIFFORD [2],[3], PETRICH [11] and LYAPIN[10]. The ideal theory in general semigroups was developed by ANJANEYULU [1]. The theory of ternary algebraic systems was introduced by LEHMER [8] in 1932. LEHMER investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semigroups are universal algebras with one associative ternary operation. The notion of ternary semigroup was known to BANACH who is credited with example of a ternary semigroup which can not reduce to a semigroup. SIOSON [15] introduced the ideal theory in ternary semigroups. He also introduced the notion of regular ternary semigroup. SANTIAGO [12] developed the theory of ternary semigroups.He studied regular and completely regular ternary semigroups.In this paper we introduce the notions of regular element, completely regular element, intra regular element in a po ternary semigroup, simple po ternary semigroupand characterize simple po ternary semigroup.

## II. Preliminaries :

**DEFINITION 2.1 :** A ternary semigroup T is said to be a ***partially ordered ternary semigroup*** if T is a partially ordered set such that $a \le b \Rightarrow [aa_1a_2] \le [ba_1a_2]$, $[a_1aa_2] \le [a_1ba_2]$, $[a_1a_2a] \le [a_1a_2b]$ for all $a, b, a_1, a_2 \in T$.

**NOTE 2.2 :** An element '***a***' of a po ternary semigroup T is a ***two sided identity*** provided *aat =att = taa =tta = t* and $t \le a$ for all $t \in$ T.

**DEFINITION 2.3**: An element *a* of a po ternary semigroup T is said to be a ***zero*** of T provided *abc = bac = bca = a*and $a \le b$ , $a \le c$ for all$b,c \in$ T.

**DEFINITION 2.4**: A nonempty subset A of a po ternary semigroup T is said to be ***poleft ternary ideal***or ***poleftideal***of T if i) $b, c \in$ T, $a \in$ A implies $bca \in$ A. ii) If $a \in$ A and $t \in$ T such that $t \le a$ then $t \in$ A.

**DEFINITION 2.5** : A nonempty subset A of a po ternary semigroup T is said to be ***polateral ternary ideal***or ***polateral ideal***of T if i) $b, c \in$ T, $a \in$ A implies $bac \in$ A. ii) If $a \in$ A and $t \in$ T such that $t \le a$ then $t \in$ A.

**DEFINITION 2.6** : A nonempty subset A of a po ternary semigroup T is said to be ***poright ternary ideal***or ***poright ideal***of T if i) $b, c \in$ T, $a \in$ A implies $abc \in$ A. ii) If $a \in$ A and $t \in$ T such that $t \le a$ then $t \in$ A.

**DEFINITION 2.7** : A nonempty subset A of a po ternary semigroup T is said to be ***potwo sided ternary ideal***or ***potwo sided ideal***of T if i) $b, c \in$ T, $a \in$ A implies $bca \in$ A, $abc \in$ A, ii) If $a \in$ A and $t \in$ T such that $t \le a$ then $t \in$ A.

**DEFINITION 2.8** : A nonempty subset A of a po ternary semigroup T is said to be ***po ternary ideal***or ***poideal***of T if i) $b, c \in$ T, $a \in$ A implies $bca \in$ A, $bac \in$ A, $abc \in$ A, ii) If $a \in$ A and $t \in$ T such that $t \le a$ then $t \in$ A.

**THEOREM 2.9 : The union of any family of poleft ideals (or lateralpoideals or right poideals or two sidedpoideals orpoideals ) of a po ternary semigroup T is a poleft ideal (or lateralpoideal or rightpoideal or two sidedpo ideal orpoideal ) of T.**

**THEOREM 2.10: The po-left ideal (orpo lateral ideals or po right ideals or po two sided ideals or po ideals )of a po ternary semigroup T generated by a non-empty subset A is the intersection of all po left ideals (or po lateral ideal or po right ideal or po two sided ideal or po ideal )of T containing A.**

**THEOREM 2.11: If T is a po ternary semigroup and $a$∈T then L($a$) = ($a$ ∪TT$a$].**

**THEOREM 2.12 : If T is a po ternary semigroup and $a$∈T then M($a$) = ($a$ ∪T$a$T∪TT$a$TT].**

**THEOREM 2.13 : If T is a po ternary semigroup and $a$∈T then R($a$) = ($a$ ∪$a$TT].**

## III. Special Elemets In Po Ternary Semigroups :

**DEFINITION 3.1 :** An element $a$ of a poternary semigroup T is said to be ***regular*** if there exist $x, y \in$ T such that $a \leq axaya$.

**NOTE 3.2:** An element $a$ of a po ternary semigroup T is regular iff $a \in (axaya]$.

**DEFINITION 3.3**:A poternary semigroup T is said to be *a* ***regular po ternary semigroup***provided every element is regular.

**DEFINITION 3.4 :** An element $a$ of a po ternary semigroup T is said to be ***left regular*** if there exist $x, y \in$ T such that $a \leq a^3xy$.

**DEFINITION 3.5 :** An element $a$ of a po-ternary semigroup T is said to be ***lateral regular*** if there exist $x, y \in$ T such that $a \leq xa^3y$.

**DEFINITION 3.6 :** An element $a$ of a po ternary semigroup T is said to be ***right regular*** if there exist $x, y \in$ T such that $a \leq xya^3$.

**DEFINITION 3.7 :** An element $a$ of a po ternary semigroup T is said to be ***intra regular*** if there exist $x, y \in$ T such that $a \leq xa^5y$.

**DEFINITION 3.8 :** An element $a$ of a po ternary semigroup T is said to be ***completely regular*** if there exist $x, y \in$ T such that $a \leq axaya$ and $axa = xaa = aax = aya = yaa = aay = axy = yxa = xay = yax$.

**DEFINITION 3.9 :** An element $a$ of a po ternary semigroup T is said to be ***completely regular*** if there exist $x, y \in$ T such that $a \in (axaya]$ and $axa = xaa = aax = aya = yaa = aay = axy = yxa = xay = yax$.

**DEFINITION 3.10 :** Apo ternary semigroup T is said to be a ***completely regularpoternary semigroup*** provided every element in T is completely regular.

**THEOREM 3.11 : Let T be a po ternary semigroup and $a \in$ T. If$a$is a completely regular element, then $a$ is regular, left regular, lateral regular and right regular.**

***Proof* :** Suppose that $a$ is completely regular.

Then there exist $x, y \in$ T such that $a \leq axaya$ and

$axa = xaa = aax = aya = yaa = aay = axy = yxa = xay = yax$. Clearly $a$ is regular.

Now $a \leq axaya \leq axaay \leq aaaxy \leq a^3xy$. Therefore $a$ is left regular.

Also $a \leq axaya \leq xaaya \leq xaaay \leq xa^3y$. Therefore $a$ is lateral regular.

And $a \leq axaya \leq xaaya \leq xyaaa \leq xya^3$. Therefore $a$ is right regular.

**THEOREM 3.12 : Let T be a po ternary semigroup and $a \in T$. If $a$ is regular then $a$ is semisimple.**

***Proof* :** Suppose that $a$ is regular.

Then $a \leq axaya$ for some $x, y \in \mathrm{T} \Rightarrow a \in (<a>^3]$

Therefore $a$ is semisimple.

**THEOREM 3.13 : Let $a$ be an element of a po ternary semigroup T. If $a$ is left regular or lateral regular or right regular, then $a$ is semisimple.**

***Proof* :**Suppose $a$ is left regular. Then $a \leq a^3xy$ for some $x, y \in \mathrm{T} \Rightarrow a \in (<a>^3]$.

Therefore $a$ is semisimple.

If $a$ is lateral regular, then $a \leq xa^3y$ for some $x, y \in$ T $\Rightarrow a \in (<a>^3]$.

Therefore $a$ is semisimple.

If $a$ is right regular, then $a \leq xya^3$ for some $x, y \in \mathrm{T} \Rightarrow a \in (<a>^3]$.

Therefore $a$ is semisimple.

**THEOREM 3.14 : Let $a$ be an element of a po ternary semigroup T. If $a$is intraregular then $a$ is semisimple.**

***Proof* :** Suppose $a$ is intra regular.

Then $a \leq xa^5y \leq xa^2a^3y$ for some $x, y \in \mathrm{T} \Rightarrow a \in (<a>^3]$.

Therefore $a$ is semisimple.

## IV. Simple Partially Ordered Ternary Semigroups :

**DEFINITION 4.1 :** A po ternary semigroup T is said to be a ***left simple poternary semigroup***if T is its only poleft ideal.

**THEOREM 4.2: Let T be a po ternarysemigroup. Then (TT*a*] is a po left ideal of T for all *a*∈T.**

***Proof* :** Let $r\in$ (TT$a$] and$b$, $c\in$T.
$r\in$ (TT$a$]$\Rightarrow r\leq x$for some$x\in$ TT$a$.
$x\in$ TT$a\Rightarrow x=yza$ where $y$, $z\in$T.
Now $r\leq x\Rightarrow bcr\leq bcx\Rightarrow bcr\leq bcyza\in$ TT$a\Rightarrow bcr\in$ (TT$a$].
Let $s\in$(TT$a$] and $t\in$T such that $t\leq s$.
$s\in$(TT$a$]$\Rightarrow s\leq x$ for some $x\in$TT$a$.
$t\leq s$, $s\leq x\Rightarrow t\leq x\Rightarrow t\in$ (TT$a$].
$t\in$T, $t\leq x$, $x\in$TT$a\Rightarrow t\in$ (TT$a$].
Hence (TT$a$] is a poleft ideal of T.

**THEOREM 4.3: A po ternary semigroup T is a left simple poternary semigroup if and only if (TT*a*] = T for all *a*∈T.**

***Proof*:** Suppose that T is a left simple po ternary semigroup and $a\in$T.
By theorem 4.2, (TT$a$] is a poleft ideal of T.
Since T is a left simple po ternary semigroup, (TT$a$] = T.
Therefore (TT$a$] = T for all $a\in$T.
Conversely suppose that (TT$a$] = T for all $a\in$T.
Let L be a poleft ideal of T.
Let $l\in$ L. Then $l\in$T. By assumption (TT$l$] = T.
Let $s\in$T. Then $s\in$ (TT$l$] $\Rightarrow s\leq xyl$ for some $x$, $y\in$T.
$l\in$ L, $x$, $y\in$T and L is a poleft ideal $\Rightarrow xyl\in$ L $\Rightarrow s\in$ L.
Therefore T ⊆ L. Clearly L ⊆T and hence T = L.
Therefore T is the only poleft ideal of T.
Hence T is left simple poternary semigroup.

**DEFINITION 4.4 :** A po ternary semigroup T is said to be a ***lateral simple poternary semigroup***if T is its only polateral ideal.

**THEOREM 4.5: Let T be a po ternarysemigroup. Then ( T*a*T∪TT*a*TT] is a polateral ideal of T for all *a*∈T.**

***Proof* :** Let $r\in$ (T$a$T∪ TT$a$TT] and$b$, $c\in$T.
$r\in$ (T$a$T∪TT$a$TT]$\Rightarrow r\leq x$ for some $x\in$T$a$T∪ TT$a$TT.
$x\in$T$a$T∪ TT$a$TT $\Rightarrow x=yaz$ or $x=yzauv$ where $y$, $z$, $u$, $v\in$T.
If $x=yaz$ then$r\leq x\Rightarrow brc\leq bxc\Rightarrow brc\leq byazc\in$TT$a$TT$\Rightarrow brc\in$ (T$a$T∪ TT$a$TT].
If $x=yzauv$ then$r\leq x\Rightarrow brc\leq bxc\Rightarrow brc\leq byzauvc\in$ T$a$T $\Rightarrow brc\in$ (T$a$T∪ TT$a$TT].
Let $s\in$(T$a$T∪ TT$a$TT] and $t\in$T such that $t\leq s$.
$s\in$(T$a$T∪ TT$a$TT]$\Rightarrow s\leq x$ for some $x\in$T$a$T∪ TT$a$TT.
$t\leq s$, $s\leq x\Rightarrow t\leq x\Rightarrow t\in$ (T$a$T∪ TT$a$TT].
$t\in$T, $t\leq x$, $x\in$T$a$T∪ TT$a$TT$\Rightarrow t\in$ (T$a$T∪ TT$a$TT].
Hence (T$a$T∪ TT$a$TT] is a polateral ideal of T.

**THEOREM 4.6: A po ternary semigroup T is a lateralsimple poternary semigroup if and only if ( T*a*T∪ TT*a*TT] = T for all *a*∈T.**

***Proof*:** Suppose that T is a lateral simple po ternary semigroup and $a\in$T.
By theorem 4.5, ( T$a$T∪ TT$a$TT] is a po lateral ideal of T.
Since T is a lateral simple poternary semigroup, ( T$a$T∪ TT$a$TT] = T.
Therefore ( T$a$T∪ TT$a$TT] = T for all $a\in$T.
Conversely suppose that ( T$a$T∪ TT$a$TT] = T for all $a\in$T.
Let M be a polateral ideal of T.
Let $m\in$M. Then $m\in$T. By assumption ( T$m$T∪ TT$m$TT] = T.
Let $s\in$T. Then $s\in$( T$m$T∪ TT$m$TT]$\Rightarrow s\leq xmy$or $s\leq xumyv$ for some $x$, $y$, $u$, $v\in$T.
$m\in$M, $x$, $y\in$T and M is a polateral ideal $\Rightarrow xmy\in$M$\Rightarrow s\in$M.
$m\in$M, $x$, $y$, $u$, $v\in$T and M is a polateral ideal $\Rightarrow xumyv\in$M$\Rightarrow s\in$M. Therefore T ⊆M. Clearly M⊆T and hence T = M.
Therefore T is the only polateral ideal of T. Hence T is lateralsimple po ternary semigroup.

**DEFINITION 4.7 :** A po ternary semigroup T is said to be a ***right simple po ternary semigroup***if T is its only poright ideal.

**THEOREM 4.8: Let T be a po ternarysemigroup. Then (*a*TT] is a poright ideal of T for all *a*∈T.**

***Proof* :** Let $r \in (aTT]$ and $b, c \in T$.
$r \in (aTT] \Rightarrow r \leq x$ for some $x \in aTT$.
$x \in aTT \Rightarrow x = ayz$ where $y, z \in T$.
Now $r \leq x \Rightarrow rbc \leq xbc \Rightarrow rbc \leq ayzbc \in aTT \Rightarrow rbc \in (aTT]$.
Let $s \in (aTT]$ and $t \in T$ such that $t \leq s$.
$s \in (aTT] \Rightarrow s \leq x$ for some $x \in aTT$.
$t \leq s,\ s \leq x \Rightarrow t \leq x \Rightarrow t \in (aTT]$.
$t \in T,\ t \leq x,\ x \in aTT \Rightarrow t \in (aTT]$.
Hence (*a*TT] is a poright ideal of T.

**THEOREM 4.9 : A po ternary semigroup T is a right simple poternary semigroup if and only if (*a*TT] = T for all *a*∈T.**

***Proof*:** Suppose that T is a right simple po ternary semigroup and $a \in T$.
By theorem 4.8, (*a*TT] is a poright ideal of T.
Since T is a right simple po ternary semigroup, (*a*TT] = T.
Therefore (*a*TT] = T for all $a \in T$.
Conversely suppose that (*a*TT] = T for all $a \in T$.
Let R be a poright ideal of T.
Let $r \in R$. Then $r \in T$. By assumption (*r*TT] = T.
Let $s \in T$. Then $s \in (rTT] \Rightarrow s \leq rxy$ for some $x, y \in T$.
$r \in R,\ x, y \in T$ and R is a poright ideal $\Rightarrow rxy \in R \Rightarrow s \in R$.
Therefore $T \subseteq R$. Clearly $R \subseteq T$ and hence T = R.
Therefore T is the only poright ideal of T.
Hence T is right simple poternary semigroup.

**DEFINITION4.10 :** An ideal A of a po ternary semigroup T is called a ***globally idempotent po-ideal*** if $(A^n] = (A]$ for all odd natural number *n*.

**DEFINITION4.11 :** A po ternary semigroup T is said to be a ***globally idempotent po ternary semigroup*** if $(T^n] = (T]$ for all odd natural number *n*.

**THEOREM 4.12** : **If A is a po ideal of a po ternary semigroup T with unity 1 and 1 ∈ A then A = T.**

***Proof*** : Clearly $A \subseteq T$.
Let $t \in T$.
$1 \in A,\ t \in T$, A is a po ideal of $T \Rightarrow 11t \in A \Rightarrow t \in A$
Therefore $T \subseteq A$.
Hence $A \subseteq T,\ T \subseteq A \Rightarrow T = A$.

**DEFINITION 4.13** : An ideal A of a po ternary semigroup T is said to be a ***proper po ideal*** of T if A is different from T.

**DEFINITION 4.14** : An ideal A of a po ternary semigroup T is said to be a ***trivial po ideal*** provided T\ A is singleton.

**DEFINITION 4.15** : An ideal A of a po ternary semigroup T is said to be a ***maximal po ideal*** provided A is a proper po ideal of T and is not properly contained in any proper po ideal of T.

**THEOREM 4.16 : If T is a po ternary semigroup with unity 1 then the union of all proper po ideals of T is the unique maximal po ideal of T.**

***Proof*** : Let M be the union of all proper po ideals of T.
Since 1 is not an element of any proper po ideal of T, $1 \notin M$.
Therefore M is a proper subset of T.
By theorem 2.9, M is a po ideal of T.
Thus M is a proper po ideal of T.
Since M contains all proper po ideals of T , M is a maximal po ideal of T.
If $M_1$ is any maximal po ideal of T, then $M_1 \subseteq M \subset T$ and hence $M_1 = M$.
Therefore M is the unique maximal po ideal of T.

**THEOREM 4.17 : In any po ternary semigroup T, the following are equivalent.**

1) **Principal po ideals of T form a chain.**
2) **Po ideals of T form a chain.**

***Proof*** : (1) $\Rightarrow$ (2): Suppose that principal po ideals of T form a chain.
Let A, B be two po ideals of T.
Suppose if possible $A \nsubseteq B$ , $B \nsubseteq A$.
Then there exist $a \in A \setminus B$ and $b \in B \setminus A$

$a \in A \Rightarrow <a> \subseteq A$ and $b \in B \Rightarrow <b> \subseteq B$.

Since principal po ideals form a chain, either $<a> \subseteq <b>$ or $<b> \subseteq <a>$ .

If $<a> \subseteq <b>$, then $a \in <b> \subseteq B$. It is a contradiction.

If $<b> \subseteq <a>$, then $b \in <a> \subseteq A$. It is also a contradiction.

Therefore $A \subseteq B$ or $B \subseteq A$ and hence po ideals form a chain.

(2) $\Rightarrow$ (1) : Suppose that po ideals of T form a chain.

Then clearly principal po ideals of T form a chain.

**DEFINITION 4.18** : A po ternary semigroup T is said to be ***simple* poternary *semigroup*** if T is its only po ideal of T.

**THEOREM 4.19 : If T is a left simple po ternary semigroup (or) a lateral simple po ternary semigroup (or) a right simple po ternary semigroup then T is a simple po ternary semigroup.**

***Proof*** : Suppose that T is a left simple po ternary semigroup.

Then T is the only poleft ideal of T.

If A is a po ideal of T, then A is a poleft ideal of T and hence A = T.

Therefore T itself is the only po ideal of T and hence T is a simple po ternary semigroup.

Suppose that T is a lateral simple po ternary semigroup.

Then T is the only polateral ideal of T.

If A is a po ideal of T, then A is a polateral ideal of T and hence A = T.

Therefore T itself is the only po ideal of T and hence T is a simple po ternary semigroup. Similarly if T is right simple poternary group then T is simple poternary semigroup.

**DEFINITION4.20** : An element $a$ of a poternary semigroup T is said to be ***semisimple*** if $a \in (<a>^3]$ i.e. $(<a>^3] = (<a>]$.

**THEOREM 4.21 :** An element $a$ of a poternary semigroup T is said to be ***semisimple*** if $a \in (<a>^n]$ i.e. $(<a>^n] = (<a>]$ for all odd natural number $n$.

***Proof* :** Suppose that $a$ is semisimple element of T.

Then $(<a>^3] = <a>$.

Let $a \in$ T and$n$ is an odd natural number.

If $n = 3$ and $a$ is semisimple then $(<a>^3] = <a>$.

If $n = 5$ then $(<a>^5] = (<a>^3](<a>^2] = (<a>](<a>^2] = (<a>^3] = (<a>]$.

Therefore by induction of $n$ is an odd natural number, we have $(<a>^n] = (<a>]$.

The converse part is trivial.

**DEFINITION4.22** : A poternary semigroup T is called ***semisimple*poternary *semigroup*** provided every element in T is semisimple.

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