

Fuzzy Optimization: Possibilistic linear equality system

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Abstract: We consider linear (and quadratic) possibilistic programs and show that the possibility distribution of their objective function remains stable under small changes in the membership functions of the fuzzy number coefficients. We generalize Kovacs's results to PLES with (Lipschitzian) fuzzy numbers and flexible linear programs.

Key words: Linear Equality System, Fuzzy Numbers, Fuzzy set of feasible solutions, Zadeh's extension principle, Kovacs's results.

I. Introduction:

The linear equality system

$$a_{i1}x_1 + \dots + a_{in}x_n = b_i, \quad i = 1, \dots, m, \quad (1)$$

or shortly,

$$Ax = b,$$

where a_{ij} , b_i and x_j are real numbers. It is known that system (1) generally belongs to the class of ill-posed problems, so a small perturbation of the parameters a_{ij} and b_i may cause a large deviation in the solution.

A possibilistic linear equality system is

$$\tilde{a}_{i1}x_1 + \dots + \tilde{a}_{in}x_n = \tilde{b}_i, \quad i = 1, \dots, m, \quad (2)$$

or shortly,

$$\tilde{A}x = \tilde{b},$$

where \tilde{a}_{ij} , $\tilde{b}_i \in F(\mathbb{R})$ are fuzzy quantities, $x \in \mathbb{R}^n$, the operations addition and multiplication by a real number of fuzzy quantities are defined by Zadeh's extension principle and the equation is understood in possibilistic sense. Recall the truth value of the assertion " \tilde{a} is equal to \tilde{b} ", written as $\tilde{a} = \tilde{b}$, denoted by Pos

($\tilde{a} = \tilde{b}$), is defined as

$$\text{Pos}(\tilde{a} = \tilde{b}) = \sup_t \{ \tilde{a}(t) \wedge \tilde{b}(t) \} = (\tilde{a} - \tilde{b})(0). \quad (3)$$

We denote by $\mu_i(x) = \text{Pos}(\tilde{a}_{i1}x_1 + \dots + \tilde{a}_{in}x_n = \tilde{b}_i)$.

Following Bellman and Zadeh the fuzzy solution (or the fuzzy set of feasible solutions) of system (2) can be viewed as the intersection of the μ_i 's such that

$$\mu(x) = \min\{\mu_1(x), \dots, \mu_m(x)\}. \quad (4)$$

A measure of consistency for the possibilistic equality system (2) is defined as

$$\mu^* = \sup\{\mu(x) \mid x \in \mathbb{R}^n\}. \quad (5)$$

Let X^* be the set of points $x \in \mathbb{R}^n$ for which $\mu(x)$ attains its maximum, if it exists. That is

$$X^* = \{x^* \in \mathbb{R}^n \mid \mu(x^*) = \mu^*\}$$

If $X^* \neq \emptyset$ and $x^* \in X^*$, then x^* is called a maximizing (or best) solution of (2).

If \tilde{a} and \tilde{b} are fuzzy numbers with $[a]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[b]^\alpha = [b_1(\alpha), b_2(\alpha)]$ then their Hausdorff distance is defined as

$$D(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \max\{|a_1(\alpha) - b_1(\alpha)|, |a_2(\alpha) - b_2(\alpha)|\}.$$

i.e. $D(\tilde{a}, \tilde{b})$ is the maximal distance between the α -level sets of \tilde{a} and \tilde{b} .

Let $L > 0$ be a real number. By $F(L)$ we denote the set of all fuzzy numbers $\tilde{a} \in F$ with membership function satisfying the Lipschitz condition with constant L , i.e.

$$|\tilde{a}(t) - \tilde{a}(t')| \leq L|t - t'|, \quad \forall t, t' \in \mathbb{R}.$$

In many important cases the fuzzy parameters \tilde{a}_{ij} , \tilde{b}_i of the system (2) are not known exactly

and we have to work with their approximations $\tilde{a}_{ij}^\delta, \tilde{b}_{ij}^\delta$ such that

$$\max_{i,j} D(\tilde{a}_{ij}^\delta, \tilde{a}_{ij}^\delta) \leq \delta. \quad \max_i D(\tilde{b}_{ij}^\delta, \tilde{b}_{ij}^\delta) \tag{6}$$

where $\delta \geq 0$ is a real number. Then we get the following system with perturbed fuzzy parameters

$$\tilde{a}_{i1}^\delta x_1 + \dots + \tilde{a}_{in}^\delta x_n = \tilde{b}_i^\delta, \quad i = 1, \dots, m \tag{7}$$

or shortly,

$$\tilde{A}_x^\delta x = \tilde{b}^\delta.$$

In a similar manner we define the solution

$$\mu^\delta(x) = \min \{ \mu_1^\delta(x), \dots, \mu_m^\delta(x) \},$$

and the measure of consistency

$$\mu^*(\delta) = \sup \{ \mu^\delta(x) \mid x \in \mathbb{R}^n \},$$

of perturbed system (7), where

$$\mu_i^\delta(x) = \text{Pos}(\tilde{a}_{i1}^\delta x_1 + \dots + \tilde{a}_{in}^\delta x_n = \tilde{b}_i^\delta)$$

denote the degree of satisfaction of the i -th equation at $x \in \mathbb{R}^n$. Let $X^*(\delta)$ denote the set of maximizing solutions of the perturbed system (7).

Kovacs showed that the fuzzy solution to the system (2) with symmetric triangular fuzzy numbers is a stable respect to small changes of centers of fuzzy parameters. Following fuller in the next theorem we establish a stability property (with respect to perturbations (6) of the solution of system (2)).

Theorem A. Let $L > 0$ and $\tilde{a}_{ij}^\delta, \tilde{b}_i, \tilde{b}_i^\delta \in F(L)$. If (6) holds, then

$$\| \mu - \mu^\delta \|_\infty = \sup_{x \in \mathbb{R}^n} | \mu(x) - \mu^\delta(x) | \leq L\delta \tag{8}$$

where $\mu(x)$ and $\mu^\delta(x)$ are the (fuzzy) solutions to systems (2) and (7), respectively.

Proof: It is sufficient to show that

$$| \mu(x) - \mu^\delta(x) | \leq L\delta$$

for each $x \in \mathbb{R}^n$ and $i = 1, \dots, m$. Let $x \in \mathbb{R}^n$ and $i \in \{1, \dots, m\}$ be arbitrarily fixed. From (3) it follows that

$$\mu_i(x) = \left(\sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i \right) (0),$$

$$\mu_i^\delta(x) = \left(\sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \right) (0).$$

Applying, Let $\tilde{a}, \tilde{b}, \tilde{c}$, and \tilde{d} be a fuzzy number. Then

$$D(\tilde{a} - \tilde{c}, \tilde{b} - \tilde{d}) \leq D(\tilde{a}, \tilde{b}) + D(\tilde{c}, \tilde{d}), \text{ we get}$$

$$D\left(\sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i, \sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta\right) \leq$$

$$\sum_{j=1}^n |x_j| D(\tilde{a}_{ij}, \tilde{a}_{ij}^\delta) + D(\tilde{b}_i, \tilde{b}_i^\delta) \leq \delta (|x|_1 + 1),$$

where $|x|_1 = |x_1| + \dots + |x_n|$. Finally, by

$$\lambda \tilde{a} \in F(L / |\lambda|), \quad \lambda \tilde{a} + \mu \tilde{b} \in F(L / |\lambda| + |\mu|)$$

if $D(\tilde{a}, \tilde{b}) \leq \delta$ then

$$\sup_t | \tilde{a}(t) - \tilde{b}(t) | \leq L\delta, \text{ we get}$$

$$\sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i \in F(L / |x|_1 + 1),$$

$$\sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta \in F(L / |x|_1 + 1)$$

and, therefore

$$\begin{aligned}
 & | \mu_i(x) - \mu_i^\delta(x) | = | (\sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i) (0) - (\sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta) (0) | \leq \\
 & \sup_t | (\sum_{j=1}^n \tilde{a}_{ij} x_j - \tilde{b}_i) (t) - (\sum_{j=1}^n \tilde{a}_{ij}^\delta x_j - \tilde{b}_i^\delta) (t) | \leq \\
 & L / |x|_1 + 1 \times \delta (|x|_1 + 1) = L\delta.
 \end{aligned}$$

Which proves the theorem.

From (8) it follows that

$$| \mu^* - \mu^*(\delta) | \leq L\delta,$$

where μ^* , $\mu^*(\delta)$ are the measures of consistency for the systems (2) and (7), respectively. It is easily checked that in the general case $\tilde{a}_{ij}, \tilde{b}_i \in F(\mathbb{R})$ the solution to possibilistic linear equality system (2) may be unstable (in metric C_∞) under small variations in the membership function of fuzzy parameters (in metric D).

To find a maximizing solution to a possibilistic linear equality system (2), then according to Negoita, we solved the following optimization problem

$$\begin{aligned}
 & \text{maximize} \quad \lambda & (9) \\
 & \mu_1(x_1, \dots, x_n) \geq \lambda, \\
 & \dots\dots\dots \\
 & \mu_m(x_1, \dots, x_n) \geq \lambda, \\
 & x \in \mathbb{R}^n, 0 \leq \lambda \leq 1.
 \end{aligned}$$

The solutions of problem (9) generally requires the use of nonlinear programming techniques, and could be tricky. However, if the fuzzy numbers in (2) are of trapezoidal form, then the problem(9) turns into a quadratically constrained programming problem.

Even though the fuzzy solution and the measure of consistency of system (2) have a stability property with respect to changes of the fuzzy parameters, the behavior of the maximizing solution towards small perturbations of the fuzzy parameters can be very fortuitous, i.e., supposing that, X^* , the set of maximizing solutions to system (2) is not empty, the distance between $x^*(\delta)$ and X^* can be very big, where $x^*(\delta)$ is a maximizing solution of the perturbed possibilistic equality system (7).

Consider the possibilistic equality system (2) with fuzzy numbers of symmetric triangular form

$$(a_{i1}, \alpha)x_1 + \dots + (a_{in}, \alpha)x_n = (b_i, \alpha), i = 1, \dots, m,$$

or shortly

$$(A, \alpha)x = (b, \alpha) \tag{10}$$

Then following Kovacs and Fuller the fuzzy solution of (10) can be written in a compact form

$$\mu(x) = \begin{cases} 1 & \text{if } Ax = b \\ 1 - \|Ax - b\|_\infty / \alpha(|x|_1 + 1) & \text{if } 0 < \|Ax - b\|_\infty < \alpha(|x|_1 + 1) \\ 0 & \text{if } \|Ax - b\|_\infty > \alpha(|x|_1 + 1) \end{cases}$$

where,

$$\|Ax - b\|_\infty = \max \{ |(a_1, x) - b_1, \dots, |(a_m, x) - b_m| \}.$$

if

$$[\mu]^1 = \{ x \in \mathbb{R}^n \mid \mu(x) = 1 \} \neq \emptyset$$

then the set of maximizing solutions, $X^* = [\mu]^1$, of (10) coincides with the solution set, denoted by X^{**} , of the crisp system $Ax = B$. The stability theorem for system (10).

Lemma 1. If

$$D(\tilde{A}, \tilde{A}^\delta) = \max_{i,j} | a_{ij} - a_{ij}^\delta | \leq \delta, D(\tilde{b}, \tilde{b}^\delta) = \max_i | b_i - b_i^\delta | \leq \delta$$

holds, then

$$\| \mu - \mu^\delta \|_\infty = \sup_x | \mu(x) - \mu^\delta(x) | \leq \delta/\alpha,$$

where $\mu(x)$ and $\mu^\delta(x)$ are the fuzzy solutions to possibilistic equality systems

$$(A, \alpha) x = (b, \alpha)$$

and,

$$(A^\delta, \alpha) x = (b^\delta, \alpha), \text{ respectively.}$$

Lemma2. Let $\tilde{a}_{ij}, \tilde{a}_{ij}^\delta, \tilde{b}_i, \tilde{b}_i^\delta \in F$ be a fuzzy numbers. If (6) holds, then

$$\| \mu - \mu^\delta \|_\infty \leq w(\delta),$$

where $w(\delta)$ denotes the maximum of modulus of continuity of all fuzzy coefficients at δ in (2) and (7).

II. Concluding Remarks:-

In this paper we have shown that the solution and the measure of consistency of the system (10) have a stability property. The behavior of the maximizing solution towards a perturbation of the fuzzy parameters has been described. Also we find Kovacs's results to PLES with (Lipschitzian) fuzzy numbers and flexible linear programs.

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