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Properties of Coxeter Andreev's Tetrahedrons

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Abstract: *Tetrahedron is the only 3-simplex convex polyhedron having four faces, and its shape has a wide application in science and technology. In this article, using graph theory and combinatorics, a study on a special type of tetrahedron called coxeter Andreev's tetrahedron has been facilitated and it has been found that there are exactly one, four and thirty coxeter Andreev's tetrahedrons having respectively two edges of order $n \geq 6$, one edge of order $n \geq 6$ and no edge of order $n \geq 6$, $n \in N$ upto symmetry.*
Keywords: *Planar graph, Dihedral angles, Coxeter tetrahedron.*
MSC 2010 Codes*: 51F15, 20F55, 51M09.*

I. Introduction

A simplex (in plural, simplexes or simplices) is a generalization [1] of the notion of a triangle or tetrahedron to arbitrary dimensions. Specifically, a *k*-simplex is a *k*-dimensional polytope which is the convex hull of its $k+1$ vertices. Tetrahedron is the only 3-simplex convex polyhedron having four faces. The angle between two faces of a polytope, measured from perpendiculars to the edge created by the intersection of the planes is called a *dihedral angle*. Roland K. W. Roeder's Theorem [11] provides the classification of compact hyperbolic tetrahedron by restricting to non-obtuse dihedral angles. A simple polytope P in n-dimensional space $X^n\left(X = E/S/H\right)$ is said to be *coxeter*, if the dihedral angles of P are of the form $\frac{\pi}{n}$ where, n is a positive integer ≥ 2. Vinberg proved in [24] that there are no compact hyperbolic coxeter polytopes in H^n when $n \geq 30$. Tumarkin classified the hyperbolic coxeter pyramids in terms of coxeter diagram and John Mcleod generalized it in his article [9]. D. A. Derevnin, at el [18] found the volume of symmetric tetrahedron.

The tetrahedron shape has a wide application [2] in engineering and computer science. Tetrahedral mess generation is one of such application. In chemistry, the tetrahedron shape is seen in nature in covalent bonds of molecules. For example, in a methane molecule $\left(CH_4\right)$ or an ammonium ion $\left(NH_4^+\right)$, four hydrogen atoms surround a central carbon or nitrogen atom with tetrahedral symmetry.

In this paper, a study on geometric shapes of a special type of tetrahedron called coxeter Andreev's tetrahedron has been carried out by the link of graph theory and combinatorics, and it has been found that there are exactly one, four and thirty coxeter Andreev's tetrahedrons having respectively two edges of order $n \geq 6$, one edge of order $n \geq 6$ and no edge of order $n \geq 6$, $n \in N$ upto symmetry.

The paper is organised as follows:
The section 1 includes introduction. The section 2 focuses some basic terminologies from graph theory and geometry. The section 3 presents new definitions and results. The conclusions are included in section 4.

II. Basic Terminologies

There is a strong link between graph theory and geometry. Graph theoretical concepts are used to understand the combinatorial structure of a polytope in geometry. Here we will mention some essential terminologies from graph theory and geometry.

Definition 2.1: A polytope is a geometric object with surfaces enclosed by edges that exist in any number of dimensions. A polytope in 2D, 3D and 4D is said to be polygon, polyhedron (plural polyhedra or polyhedrons) and polychoron respectively. The enclosed surfaces are said to be faces. The line of intersection of any two faces is said to be an edge and a point of intersection of three or more edges is called a vertex.

Definition 2.2: Let P be a polyhedron. The abstract graph of P is denoted by $G(P)$ and is defined as $G(P)=(V(P),E(P))$, where $V(P)$ is the set of vertices of P and two vertices $x,y\in V(P)$ are adjacent if and only if (x,y) is an edge of P.

Definition 2.3: A *coxeter* dihedral angle is a dihedral angle of the form $\frac{\pi}{n}$ where, n is a positive integer ≥ 2. A polytope with coxeter dihedral angles is called a *coxeter polytope*.

Remark 2.4: A non-obtuse angle α is such that $0<\alpha\leq\frac{\pi}{2}$. Coxeter dihedral angles are of the form $\frac{\pi}{n}$, n is a positive integer ≥ 2. Therefore coxeter dihedral angles are non-obtuse.

Definition 2.5: A prismatic $k-$circuit $\Gamma_p(k)$ is a $k-$circuit such that no two edges of C which correspond to edges traversed by $\Gamma_p(k)$ share a common vertex.

Definition 2.6: A cell complex C on S^2 is called trivalent if each vertex is the intersection of three faces.

Definition 2.7: A 3-dimensional combinatorial polytope is a cell complex C on S^2 that satisfies the following conditions:

(a) Each edge of C is the intersection of exactly two faces
(b) A nonempty intersection of two faces is either an edge or a vertex.
(c) Each face is enclosed by not less than 3 edges.

Any trivalent cell complex C on S^2 that satisfies the above three conditions is said to be *abstract polyhedron*.

Definition 2.8: A 3D polytope is called *simplicial* if every face contains exactly 3 vertices. A 3D polytope is called a *simple polytope* if each vertex is the intersection of exactly 3 faces.

The 1-skeleton of a polytope is the set of vertices and edges of the polytope. The skeleton of any convex polyhedron is a planar graph and the skeleton of any $k-$dimensional convex polytope is a $k-$connected graph.

Theorem 2.9: (Blind and Mani) *If P is a convex polyhedron, then the graph $G(P)$ determines the entire combinatorial structure of P. In other words, if two simple polyhedral have isomorphic graphs, then their combinatorial polyhedral are also isomorphic.*

Theorem 2.10: (Ernst Steinitz) *A graph $G(P)$ is a graph of a 3-dimensional polytope P if and only if it is simple, planar and 3-connected.*

Corollary 2.11: *Every 3-connected planar graph can be represented in a plane such that all the edges are straight lines, all the bounded regions determined by these and the union of all the bounded regions are convex polygons.*

III. New Definitions and Results

Definition 3.1: *If the dihedral angle of an edge of a polytope is $\frac{\pi}{n}, n$ is a positive number, then n is said to be the order of the edge. We define a trivalent vertex to be of order (l,m,n) if the three edges at that vertex are of order l,m,n.*

Definition 3.2: *An Andreev's polytope is an abstract polytope which satisfies the following Andreev's conditions [16].*

(1) *Each dihedral angle* α_i *is non-obtuse* $\left(0 < \alpha_i \le \frac{\pi}{2}\right)$.

(2) *Whenever three distinct edges* e_i, e_j, e_k *meet at a vertex, then* $\alpha_i + \alpha_j + \alpha_k > \pi$.

(3) *Whenever* $\Gamma_p(3)$ *intersecting edges* e_i, e_j, e_k*, then* $\alpha_i + \alpha_j + \alpha_k < \pi$.

(4) *Whenever* $\Gamma_p(4)$ *intersecting edges* e_i, e_j, e_k, e_l*, then* $\alpha_i + \alpha_j + \alpha_k + \alpha_l < 2\pi$.

(5) *Whenever there is a four sided face bounded by edges* e_1, e_2, e_3, e_4*, enumerated successively, with edges* $e_{12}, e_{23}, e_{34}, e_{41}$ *entering the four vertices (edge* e_{ij} *connects to the ends of* e_i *and* e_j*), then* $\alpha_1 + \alpha_3 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi$*, and* $\alpha_2 + \alpha_4 + \alpha_{12} + \alpha_{23} + \alpha_{34} + \alpha_{41} < 3\pi$.

An Andreev's polytope with coxeter dihedral angles is called a coxeter Andreev's polytope. If the Andreev's polytope is not simplex, then it can be realized in Hyperbolic space [12, 16].

In our work, we pursue the coxeter Andreev's tetrahedron which is a simplex having no prismatic $k-$circuit $\Gamma_p(k)$, no four sided face, but its dihedral angles are non-obtuse.

Corollary 3.3: *In a coxeter Andreev's tetrahedron* T*, the order of each vertex is one of the forms:* $(2,2,n \ge 2),(2,3,3),(2,3,4),(2,3,5)$.

Proof: Suppose the order of one vertex of a coxeter Andreev's tetrahedron T is (n_i, n_j, n_k). By second condition of Andreev's polytope:

$$\frac{\pi}{n_i} + \frac{\pi}{n_j} + \frac{\pi}{n_k} > \pi \Rightarrow \frac{1}{n_i} + \frac{1}{n_j} + \frac{1}{n_k} > 1$$

So, upto permutations, the triples (n_i, n_j, n_k) are respectively $(2,2,n \ge 2),(2,3,3),(2,3,4)$, $(2,3,5)$.

Remarks 3.4:

- *Suppose the dihedral angles at the edges* $e_1, e_2, e_3, e_4, e_5, e_6$ *of a coxeter Andreev's tetrahedron are respectively* $\frac{\pi}{n_1}, \frac{\pi}{n_2}, \frac{\pi}{n_3}, \frac{\pi}{n_4}, \frac{\pi}{n_5}, \frac{\pi}{n_6}$ *as shown in figure 3.1.*

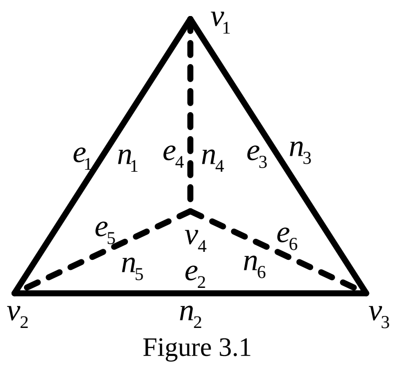


Figure 3.1

Then we denote the coxeter Andreev's tetrahedron as $T = [n_1, n_2, n_3, n_4, n_5, n_6]$.

- *We use the notation* $\mathrm{T_{kn}}$-j *to denote the jth coxeter Andreev's tetrahedron with* k *number of edges of order* $n \geq 6$.
- *For our convenient, we split* $(2,2,n\geq 2),(2,3,3),(2,3,4),(2,3,5)$ *as:* $(2,2,n\geq 6)$, $(2,2,2),(2,2,3)$, $(2,2,4),(2,2,5),(2,3,3),(2,3,4),(2,3,5)$.

Theorem 3.5: *Let* f_i *and* f_j *be two distinct triangular faces of an abstract polyhedron* T. *Then* T *is a tetrahedron if and only if* $f_i \cap f_j \neq \phi$.

Proof: If T is a tetrahedron then $f_i \cap f_j$ gives either an edge or a vertex, therefore, $f_i \cap f_j \neq \phi$. Conversely, suppose $f_i \cap f_j \neq \phi$. Since T is an abstract polyhedron, therefore T is trivalent, that is, the degree of each vertex is 3. Let $f_1=\{v_1,v_2,v_3\}$, $f_2=\{v_1,v_2,v_4\}$, $f_3=\{v_2,v_3,v_4\}$ and $f_4=\{v_1,v_3,v_4\}$ as shown in the figure 3.2.

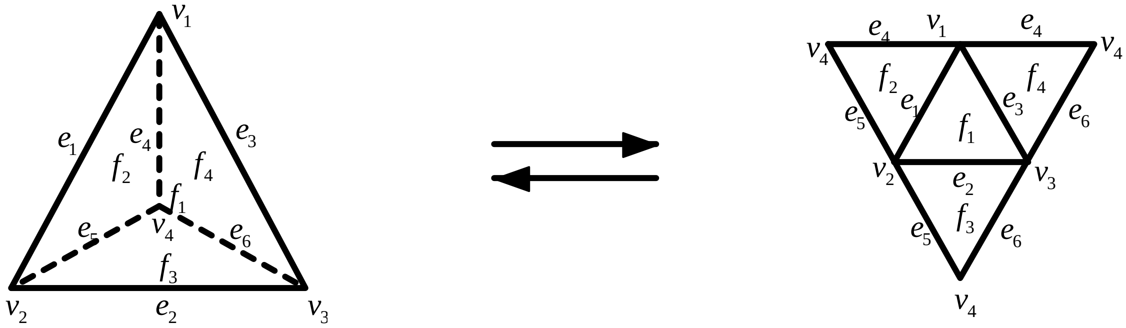


Figure 3.2

Now,

$$f_1\cap f_2=\{e_1=[v_1,v_2]\},\ f_1\cap f_3=\{e_2=[v_2,v_3]\},\ f_1\cap f_4=\{e_3=[v_1,v_3]\}$$
$$f_2\cap f_3=\{e_5=[v_2,v_4]\},\ f_2\cap f_4=\{e_4=[v_1,v_4]\},\ f_3\cap f_4=\{e_6=[v_3,v_4]\}$$

And

$$f_1\cap f_2\cap f_4=\{v_1\},\ f_1\cap f_2\cap f_3=\{v_2\},\ f_1\cap f_3\cap f_4=\{v_3\},\ f_2\cap f_3\cap f_4=\{v_4\}$$

Since, $f_i \cap f_j$ is an edge for any two faces f_i, f_j and $f_i\cap f_j\cap f_k$ is a vertex for any three faces f_i, f_j, f_k. Therefore P is a tetrahedron.

Corollary 3.6: *In a coxeter Andreev's tetrahedron* T, *the number of edges of order 2 at one vertex is at least 1 and at most 3.*

Proof: Clear from corollary 3.3.

Corollary 3.7: *In a coxeter Andreev's tetrahedron* T, *the edges of order* $n \geq 6$ *are disjoint.*

Proof: Suppose the edges of order $n\geq 6$ are not disjoint. Then, there exists at least two adjacent edges e_i, e_j at one vertex v with orders $n_i, n_j \geq 6$. Let e_k be another edge at vertex v. By corollary 3.6, the order of e_k is 2 which is adjacent to the edges e_i, e_j.

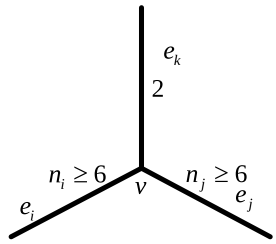


Figure 3.3

Using second condition of Andreev's polytope:

$$\frac{\pi}{n_i} + \frac{\pi}{n_j} + \frac{\pi}{2} > \pi \qquad (1)$$

Since $n_i, n_j \geq 6$, therefore,

$$\frac{\pi}{n_i} + \frac{\pi}{n_j} + \frac{\pi}{2} \leq \frac{\pi}{6} + \frac{\pi}{6} + \frac{\pi}{2} = \frac{5\pi}{6} < \pi$$

This is contradiction to (1). Therefore, the edges of order $n \geq 6$ are disjoint.

Corollary 3.8: *In a coxeter Andreev's tetrahedron T, if an edge at one vertex is of order $n \geq 6$, then the other two edges must be of order 2.*

Proof: Let e_i, e_j, e_k be three edges at one vertex v with orders $n_i \geq 6, n_j, n_k$ respectively.

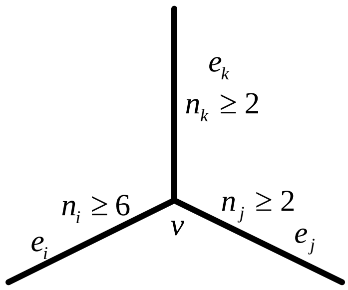


Figure 3.4

Using second condition of Andreev's polytope:

$$\frac{\pi}{n_i} + \frac{\pi}{n_j} + \frac{\pi}{n_k} > \pi$$

Since $n_i \geq 6$, therefore:

$$\pi < \frac{\pi}{n_i} + \frac{\pi}{n_j} + \frac{\pi}{n_k} \Rightarrow \pi < \frac{\pi}{6} + \frac{\pi}{n_j} + \frac{\pi}{n_k} \Rightarrow \frac{5\pi}{6} < \frac{\pi}{n_j} + \frac{\pi}{n_k} \qquad (2)$$

Since, n_j, n_k are positive integers, therefore the inequality (3) has only the solutions $n_j = n_k = 2$.

Corollary 3.9: *In a coxeter Andreev's tetrahedron T, there exists at most one edge at one vertex of order $n \geq 6$.*

Proof: Suppose, there exists at least two edges at one vertex whose orders are $n \geq 6$. But by corollary 3.7, the edges of order $n \geq 6$ are disjoint. Hence, our assumption is false and therefore, there exists at most one edge at one vertex of order $n \geq 6$.

Corollary 3.10: *In a coxeter Andreev's tetrahedron T, there are at most two edges are of order $n \geq 6$.*

Proof: In a tetrahedron T, there are exactly two disjoint edges upto symmetry. By corollary 3.7, the edges of order $n \geq 6$ are disjoint. Therefore there are at most two edges of order $n \geq 6$.

Theorem 3.11: *In a tetrahedron T, if any three vertices are of same order, then the fourth vertex is also of same order.*

Proof: Suppose any three vertices v_1, v_2 and v_3 of a tetrahedron T are of same order (n_1, n_2, n_3) up to symmetry. It is well known that, in a tetrahedron, any two vertices are adjacent to each other. Therefore, v_1, v_2 and v_3 are adjacent to v_4 and suppose, they are adjacent to v_4 by the edges of order n_2, n_3 and n_1 respectively as shown in the figure 3.5.

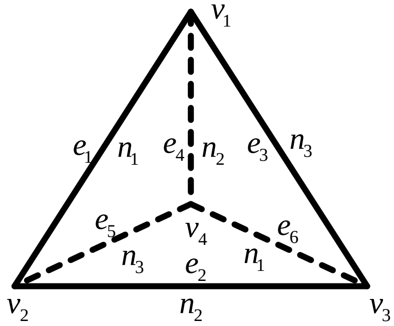


Figure 3.5

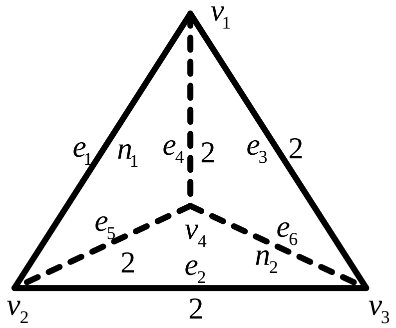
Then, the order of v_4 is (n_1, n_2, n_3) up to symmetry.

Corollary 3.12: *In a tetrahedron T, the number of same order vertices can be either 2 or 4.*

Proof: It is obvious that two vertices in a tetrahedron T can have same order. Again by Theorem 3.11, if any three vertices are of same order, then the fourth vertex is also of same order. That is, there cannot be exactly three vertices of same order. Therefore, in a tetrahedron T, the number of same order vertices can be either 2 or 4.

Theorem 3.13: *In a coxeter Andreev's tetrahedron T, if exactly two edges are of order $n \geq 6$, then there exists exactly 1 such T upto symmetry.*

Proof: Let T be a coxeter Andreev's tetrahedron. Also let T has exactly two edges (disjoint, by corollary 3.7) e_1 and e_6 are of orders $n_1, n_2 \geq 6$. To avoid symmetry, let us assume $n_1 \leq n_2$. Now, since any one of the remaining edges e_2, e_3, e_4, e_5 is adjacent to one of the edges e_1 and e_6. By corollary 3.8, if an edge at one vertex is of order $n \geq 6$, then the other two edges must be of order 2. Therefore, we have only the choice that the remaining edges e_2, e_3, e_4, e_5 are of order 2. Hence, there is exactly 1 such tetrahedron T upto symmetry with two edges of orders $n \geq 6$.



$T_{2n}\text{-}1 = [n_1 \geq 6, 2, 2, 2, 2, n_2 \geq 6]$

Figure 3.6

Note: For different values of n_1 and n_2, there will be infinite numbers of tetrahedrons, but we treat these as single category (coxeter Andreev's tetrahedron with exactly two edges are of order $n \geq 6$).

Theorem 3.14: *In a coxeter Andreev's tetrahedron T, if exactly one edge is of order $n \geq 6$, then there exists exactly 4 such T upto symmetry.*

Proof: Let T be a coxeter Andreev's tetrahedron. Also let T has exactly one edge e_1 is of order $n \geq 6$. By corollary 3.8, if an edge at one vertex is of order $n \geq 6$, then the other two edges must be of order 2. Therefore, each of the edges e_2, e_3, e_4, e_5 must be of order 2.

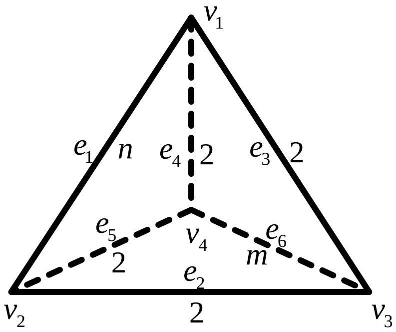
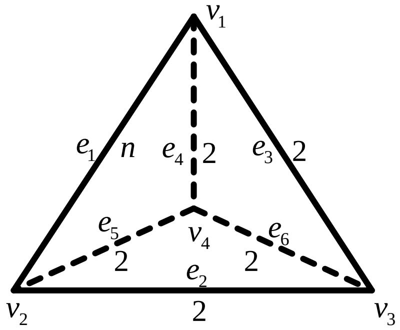
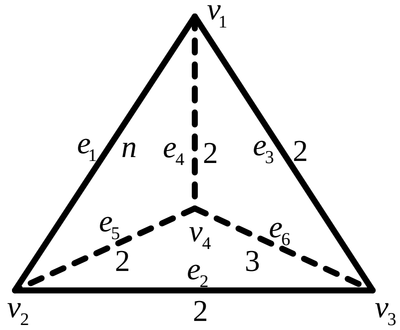


Figure 3.7

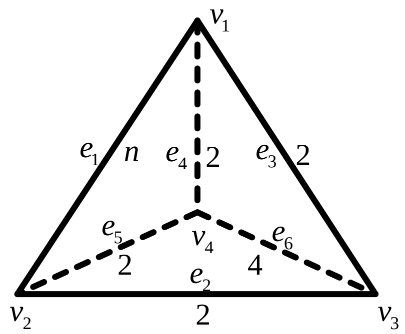
Now, the choices of the orders of the remaining edge e_6 are $m = 2, 3, 4, 5$. Therefore, there are exactly 4 such T upto symmetry with exactly one edge of order $n \geq 6$.



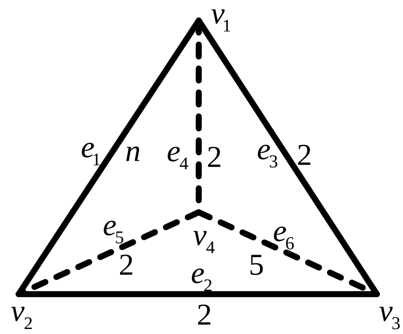
$T_{1n}\text{-}1 = [n \geq 6, 2, 2, 2, 2, 2]$



$T_{1n}\text{-}2 = [n \geq 6, 2, 2, 2, 2, 3]$



$T_{1n}\text{-}3 = [n \geq 6, 2, 2, 2, 2, 4]$



$T_{1n}\text{-}4 = [n \geq 6, 2, 2, 2, 2, 5]$

Figure 3.8

Note: For different values of n, there will be infinite numbers of tetrahedrons, but we treat these as single category (coxeter Andreev's tetrahedron with exactly one edge is of order $n \geq 6$).

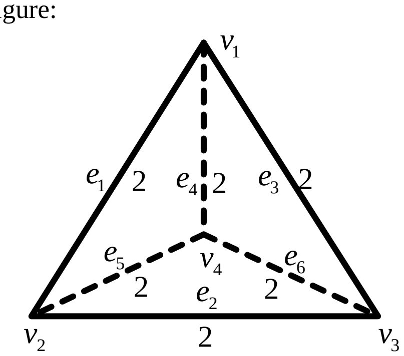
Theorem 3.15: *In a coxeter Andreev's tetrahedron* T. *If* T *has no edge of order* $n \geq 6$*, then there are exactly 10 such* T *upto symmetry with at least one vertex is of order* $(2,2,2)$.

Proof: Let T be a coxeter Andreev's tetrahedron. Also let T has no edge of order $n \geq 6$ and with at least one vertex is of order $(2,2,2)$. By corollary 3.12, the number of same order vertices can be either 2 or 4.

Therefore, there will be three cases: Case 1: all (four) the vertices are of order $(2,2,2)$, Case 2: two vertices are of order $(2,2,2)$ and Case 3: one vertex is of order $(2,2,2)$.

Case 1: All the vertices of T are of order $(2,2,2)$.

In this case, we have the following figure:



$T_{0n}\text{-}1 = [2, 2, 2, 2, 2, 2]$

Figure 3.9

Therefore, there is exactly 1 T of this type up to symmetry with all the vertices of order $(2,2,2)$.

Case 2: Two vertices are of order $(2,2,2)$.

Suppose, the two vertices v_1 and v_2 are of order $(2,2,2)$ as shown in the figure:

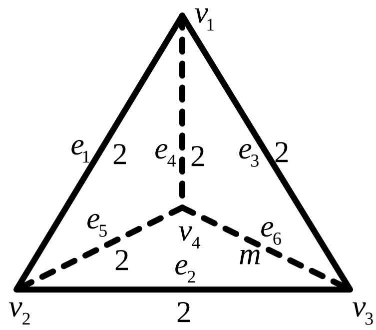


Figure 3.10

Then, the choices of the order of the edge e_6 will be $m=2,3,4,5$. For $m=2$, it falls in case 1. For $m=3,4,5$, there are exactly 3 such T upto symmetry.

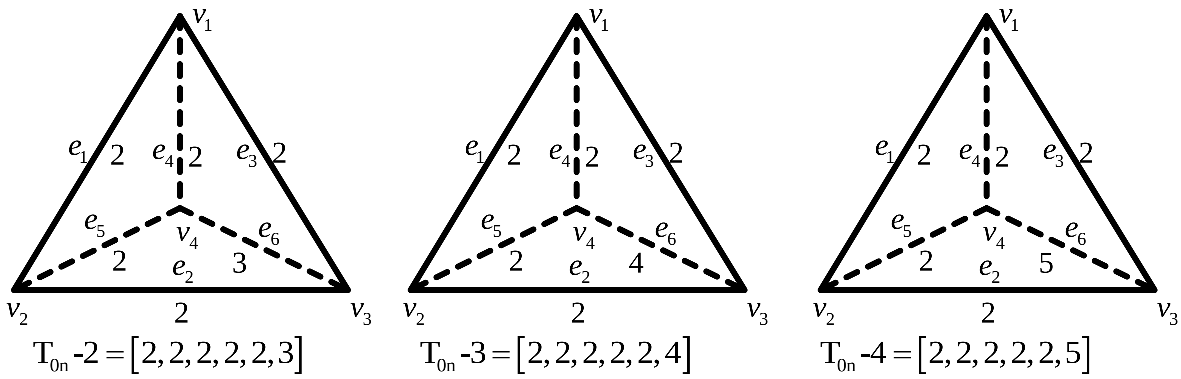


Figure 3.11

Case 3: One vertex v_1 is of order $(2,2,2)$.

In this case, we can order at most one edge out of e_2, e_5, e_6 with 2 because, if we order two or more (all) edges of e_2, e_5, e_6 by 2, then it falls in case 2 or case 1 respectively.

Case 3.1: At most one e_2 is of order 2.

Suppose e_2, e_5, e_6 are ordered by 2, m_1, and m_2 respectively. The possible choices for $m_1, m_2 = 3,4,5$.

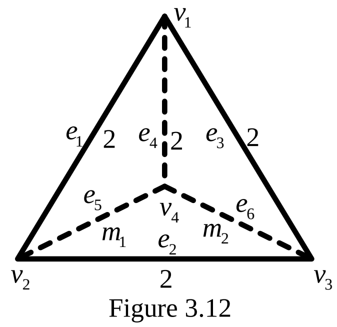
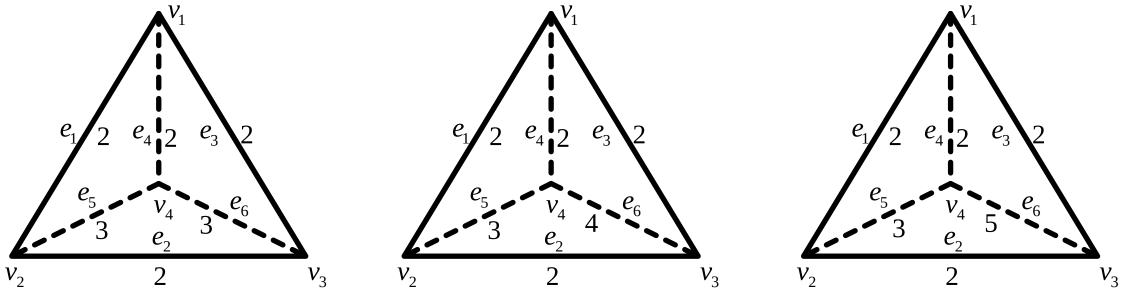


Figure 3.12

To avoid symmetry, assume $m_1 \le m_2$. Therefore $m_1 = 3$ and $m_2 = 3,4,5$ upto symmetry. Hence, there are exactly 3 such T upto symmetry.



T_{0n}-5 = [2,2,2,2,3,3] T_{0n}-6 = [2,2,2,2,3,4] T_{0n}-7 = [2,2,2,2,3,5]

Figure 3.13

Case 3.2: No one of the edges e_2, e_5, e_6 is ordered by 2.

Suppose, e_2, e_5, e_6 are ordered by m_1, m_2 and m_3 respectively. The possible choices for $m_1, m_2, m_3 = 3, 4, 5$.

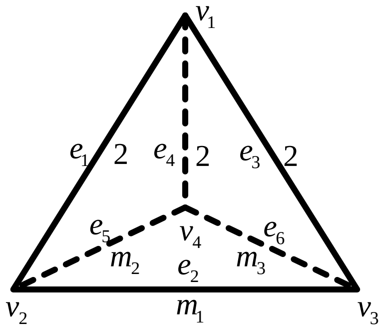
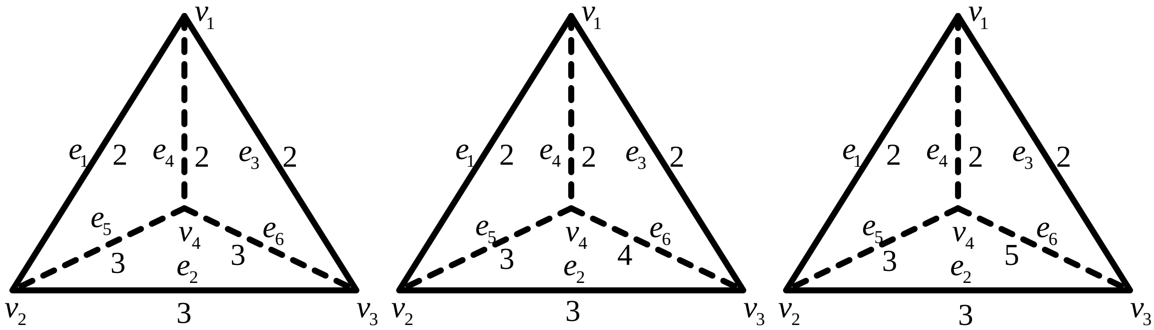


Figure 3.14

To avoid symmetry, let us assume $m_1 \le m_2 \le m_3$. Therefore, the orders of (e_2, e_5, e_6) are $(3,3,3), (3,3,4), (3,3,5)$ upto symmetry. Hence, there are exactly 3 such T upto symmetry.



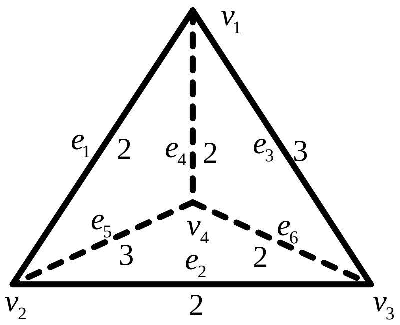
T_{0n}-8 = [2,2,2,3,3,3] T_{0n}-9 = [2,2,2,3,3,4] T_{0n}-10 = [2,2,2,3,3,5]

Figure 3.15

Theorem 3.16: *In a coxeter Andreev's tetrahedron T. If T has no edge of order $n \ge 6$, then there are exactly 8 such T upto symmetry with at least one vertex is of order $(2,2,3)$ and no vertex is of order $(2,2,2)$.*

Proof: Let T be a coxeter Andreev's tetrahedron. Also let T has no edge of order $n \ge 6$, at least one vertex is of order $(2,2,3)$ and no vertex is of order $(2,2,2)$. By corollary 3.12, the number of same order vertices can be either 2 or 4. Therefore, there will be three cases: Case 1: all (four) the vertices are of order $(2,2,3)$, Case 2: two vertices are of order $(2,2,3)$ and Case 3: one vertex is of order $(2,2,3)$.

Case 1: All the vertices of T are of order $(2,2,3)$.
In this case, we have the following figure:



$\mathrm{T_{0n}}\text{-}11 = [2,2,3,2,3,2]$

Figure 3.16

Therefore, there are exactly 1 such T upto symmetry.

Case 2: Two vertices are of order $(2,2,3)$.

In this case, we can have the two vertices of order $(2,2,3)$ with either adjacent edges of order 3 or disjoint edges of order 3 or they share a common edge of order 3.

Case 2.1: Two vertices of order $(2,2,3)$ with adjacent edges of order 3.

Suppose, the vertices v_1 and v_2 are ordered by $(2,2,3)$, where the edges e_2 and e_3 of order 3 are adjacent. Then, the only possibility for e_6 is of order 2 and then, the order of v_4 becomes $(2,2,2)$, which cannot be taken by assumption.

Case 2.2: Two vertices of order $(2,2,3)$ with disjoint edges of order 3

Suppose, the vertices v_1 and v_2 are ordered by $(2,2,3)$, where the edges e_3 and e_5 of order 3 are disjoint. Then, the possibilities of orders for e_6 are $m = 2,3,4,5$.

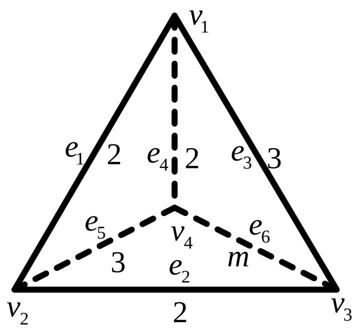
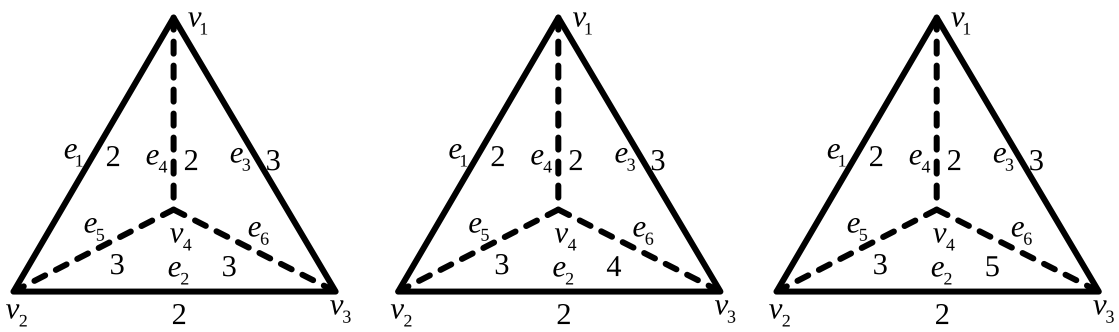


Figure 3.17

For $m = 2$, it falls in case 1. For $m = 3,4,5$, the number of T of this type is exactly 3 up to symmetry.



$T_{0n}\text{-}12=[2,2,3,2,3,3]$ $T_{0n}\text{-}13=[2,2,3,2,3,4]$ $T_{0n}\text{-}14=[2,2,3,2,3,5]$

Figure 3.18

Case 2.3: Two vertices of order $(2,2,3)$ with common edge of order 3.

Suppose the vertices v_1 and v_3 are ordered by $(2,2,3)$ sharing the common edge e_3 of order 3. Then, the possibilities of orders for e_5 are $m=2,3,4,5$.

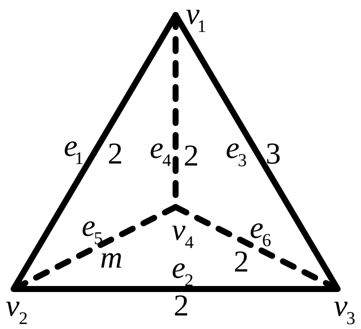
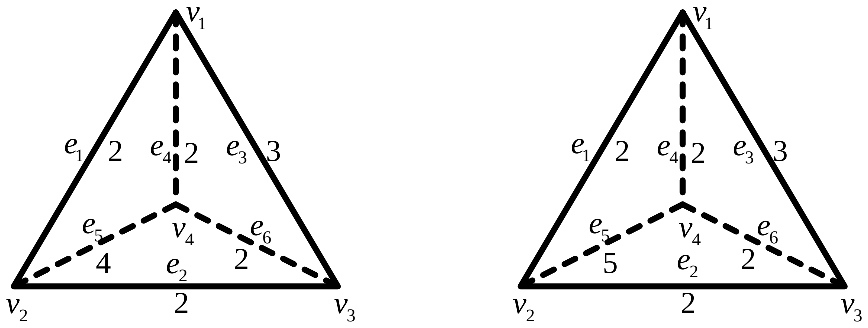


Figure 3.19

For $m=2$, the order of v_2 becomes $(2,2,2)$, which cannot be taken by assumption.
For $m=3$, it falls in case 1.
For $m=4,5$, there are exactly 2 T of this type upto symmetry



$T_{0n}\text{-}15=[2,2,3,2,4,2]$ $T_{0n}\text{-}16=[2,2,3,2,5,2]$

Figure 3.20

Case 3: One vertex v_1 is of order $(2,2,3)$.

By corollary 3.6, the number of edges of order 2 at one vertex is at least 1 and at most 3. Therefore, at v_3, there exists at least one edge e_2 of order 2 upto symmetry. If e_6 is also of order 2, then it falls in case 2. Suppose e_5 and e_6 are ordered by m_1 and m_2 respectively.

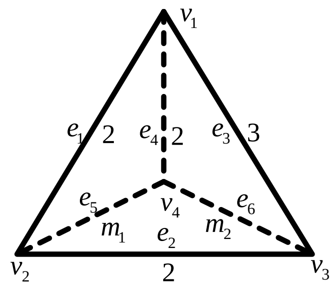
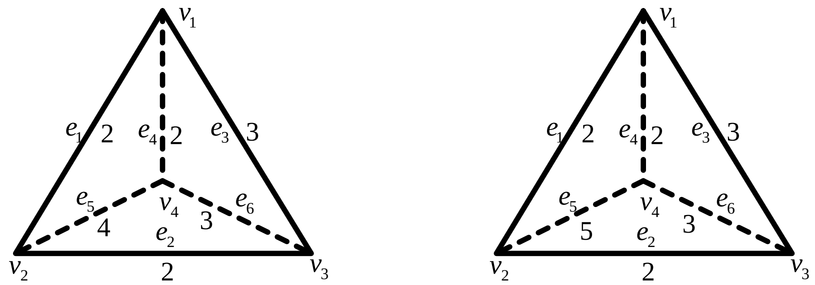


Figure 3.21

Then, the possibilities for $m_1 = 2,3,4,5$. If $m_1 = 2$, the order of v_2 becomes $(2,2,2)$, which cannot be taken by assumption. For $m_1 = 3$, it falls in case 2. Therefore, $m_1 = 4,5$.

Now, the possibilities for $m_2 = 3,4,5$. For $m_2 = 3$, we have exactly 2 such T upto symmetry.



$T_{0n}\text{-}17 = [2,2,3,2,4,3]$ $T_{0n}\text{-}18 = [2,2,3,2,5,3]$

Figure 3.22

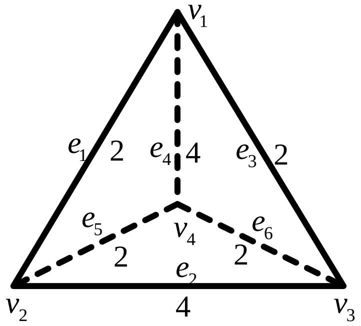
For $m_2 = 4,5$, the order of the vertex v_4 becomes $(2,4,4),(2,4,5),(2,5,5)$ respectively, which are not possible by corollary 3.3.

Theorem 3.17: *In a coxeter Andreev's tetrahedron T. If T has no edge of order $n \geq 6$, then there are exactly 4 such T upto symmetry with at least one vertex is of order $(2,2,4)$ and no vertex is of order of the forms $(2,2,2),(2,2,3)$.*

Proof: Let T be a coxeter Andreev's tetrahedron. Also let T has no edge of order $n \geq 6$, at least one vertex is of order $(2,2,4)$ and no vertex is of order $(2,2,2),(2,2,3)$. By corollary 3.12, the number of same order vertices can be either 2 or 4. Therefore, there will be three cases: Case 1: all (four) the vertices are of order $(2,2,4)$, Case 2: two vertices are of order $(2,2,4)$ and Case 3: one vertex is of order $(2,2,4)$.

Case 1: All the vertices of T are of order $(2,2,4)$.

In this case, there are exactly 1 T of this type upto symmetry.



$T_{0n}\text{-}19 = [2,4,2,4,2,2]$

Figure 3.23

Case 2: Two vertices are of order $(2,2,4)$.

If the two vertices of order $(2,2,4)$ are with adjacent edges of order 4, then the order of the vertex at which the two edges of order 4 are adjacent becomes $(n \geq 2,4,4)$. This is not possible as it is not in coxeter Andreev's tetrahedron. Therefore, the two vertices of order $(2,2,4)$ cannot be with adjacent edges of order 4 and hence, we can have the two vertices of order $(2,2,4)$ with either disjoint edges of order 4 or common edge of order 4.

Case 2.1: Two vertices of order $(2,2,4)$ with disjoint edges of order 4.

Suppose the vertices v_1 and v_2 are of order $(2,2,4)$ with disjoint edges e_3 and e_5 of order 4.

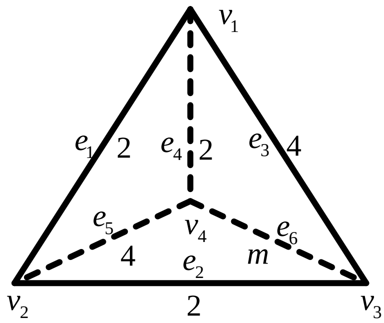
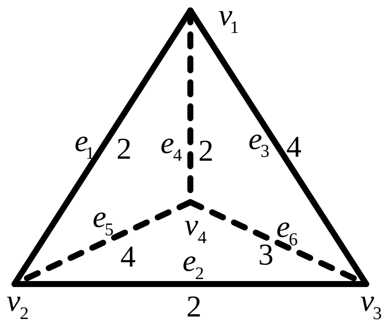


Figure 3.24

Then, the possibilities of orders for e_6 are $m = 2,3,4,5$. For $m = 2$, it falls in case 1. For $m = 3$, we have exactly 1 T of this type upto symmetry.



$T_{0n}\text{-}20 = [2,2,4,2,4,3]$

Figure 3.25

For $m = 4,5$, the order of v_3 becomes $(2,4,4),(2,4,5)$ respectively which are not in coxeter Andreev's tetrahedron.

Case 2.2: Two vertices of order $(2,2,4)$ with common edge of order 4.

Suppose the vertices v_1 and v_3 are of order $(2,2,4)$ with common edge e_3 of order 4.

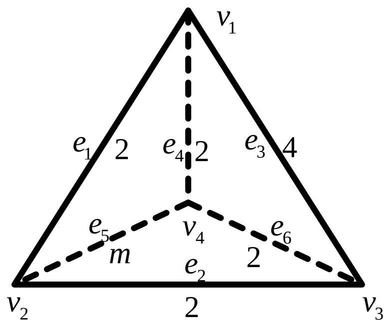
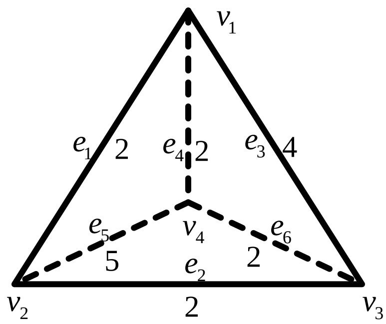


Figure 3.26

Then the possibilities of order for e_5 are $m = 2,3,4,5$. For $m = 2,3$, the order of v_2 becomes $(2,2,2),(2,2,3)$ respectively which are not taken by assumption. For $m = 4$, the order of v_2 becomes $(2,2,4)$ and it falls in case 1. For $m = 5$, we have exactly 1 T of this type upto symmetry.



T_{0n}-21 $= [2,2,4,2,5,2]$

Figure 3.27

Case 3: One vertex v_1 is of order $(2,2,4)$.

By corollary 3.6, the number of edges of order 2 at one vertex is at least 1 and at most 3. Therefore, at v_3, there exists at least one edge e_2 of order 2 upto symmetry. If e_6 is also of order 2, then it falls in case 2.

Suppose e_5 and e_6 are ordered by m_1 and m_2 respectively.

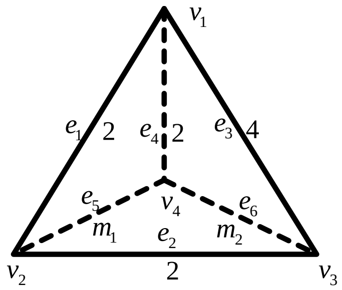
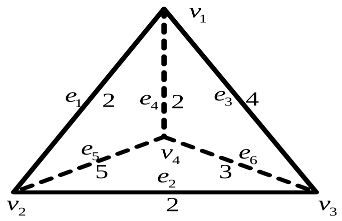


Figure 3.28

Then the possibilities for $m_1 = 2,3,4,5$. If $m_1 = 2,3$, the order of v_2 becomes $(2,2,2)$ and $(2,2,3)$ respectively, which are not taken by assumption. For $m_1 = 4$, it falls in case 2. Therefore $m_1 = 5$. Again, the possibilities for $m_2 = 2,3,4,5$. For $m_2 = 2$, it falls in case 2. Therefore $m_2 = 3$ and hence there are exactly 1 CHC tetrahedron T upto symmetry.



T_{0n}-22 $= [2,2,4,2,5,3]$

Figure 3.29

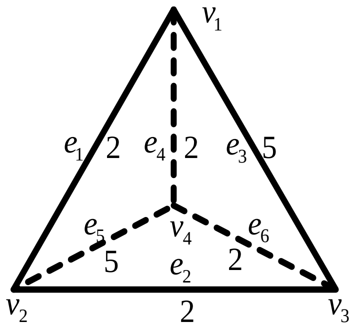
For $m_2 = 4,5$, the order of the vertex v_4 becomes $(2,4,5),(2,5,5)$ respectively, which are not possible by corollary 3.3.

Theorem 3.18: *In a coxeter Andreev's tetrahedron T. If T has no edge of order $n \geq 6$, then there are exactly 2 such T upto symmetry with at least one vertex is of order $(2,2,5)$ and no vertex is of order of the forms $(2,2,2),(2,2,3),(2,2,4)$.*

Proof: Let T be a coxeter Andreev's tetrahedron. Also let T has no edge of order $n \geq 6$, at least one vertex is of order $(2,2,5)$ and no vertex is of order $(2,2,2),(2,2,3),(2,2,4)$. By corollary 3.12, the number of same order vertices can be either 2 or 4. Therefore, there will be three cases: Case 1: all (four) the vertices are of order $(2,2,5)$, Case 2: two vertices are of order $(2,2,5)$ and Case 3: one vertex is of order $(2,2,5)$.

Case 1: All the vertices of T are of order $(2,2,5)$.

In this case, we have exactly 1 T of this type upto symmetry.



T_{0n}-23 $= [2,2,5,2,5,2]$

Figure 3.30

Case 2: Two vertices are of order $(2,2,5)$.

If the two vertices of order $(2,2,5)$ are with adjacent edges of order 5, then the order of the vertex at which the two edges of order 5 are adjacent becomes $(n \geq 2,5,5)$. This is not possible by corollary 3.3. Therefore, the two vertices of order $(2,2,5)$ cannot be with adjacent edges of order 5 and hence, we can have the two vertices of order $(2,2,5)$ with either disjoint edges of order 5 or common edge of order 5.

Case 2.1: Two vertices of order $(2,2,5)$ with disjoint edges of order 5.

Suppose the vertices v_1 and v_2 are of order $(2,2,5)$ with disjoint edges e_3 and e_5 of order 5. Then, the possibilities of orders for e_6 are $m = 2,3,4,5$.

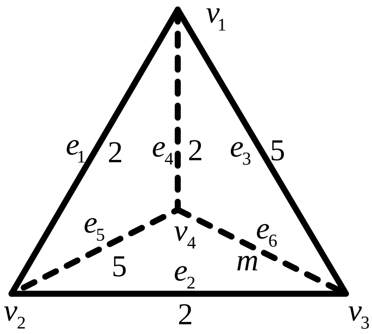
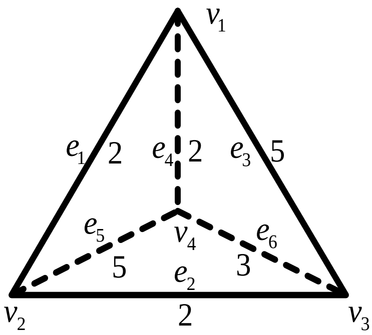


Figure 3.31

For $m = 2$, it falls in case 1. For $m = 3$, we have exactly 1 T of this type upto symmetry.



$T_{0n}\text{-}24 = [2, 2, 5, 2, 5, 3]$

Figure 3.32

For $m = 4, 5$, the order of v_3 becomes $(2, 4, 5), (2, 5, 5)$ respectively which are not possible by corollary 3.3.

Case 2.2: Two vertices of order $(2, 2, 5)$ with common edge of order 5.

Suppose the vertices v_1 and v_3 are of order $(2, 2, 5)$ with common edge e_3 of order 5.

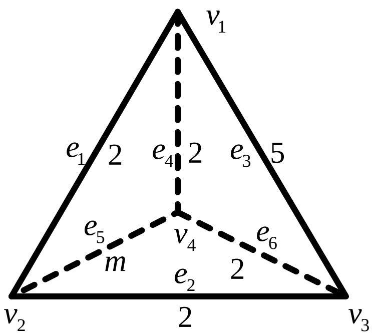


Figure 3.33

Then the possibilities of order for e_5 are $m = 2, 3, 4, 5$. For $m = 2, 3, 4$, the order of v_2 becomes $(2, 2, 2), (2, 2, 3), (2, 2, 4)$ respectively which are not taken by assumption. For $m = 5$, it falls in case 1.

Case 3: One vertex v_1 is of order $(2, 2, 5)$.

By corollary 3.6, the number of edges of order 2 at one vertex is at least 1 and at most 3. Therefore, at v_3, there exists at least one edge e_2 of order 2 upto symmetry. If e_6 is also of order 2, then it falls in case 2.
Suppose e_5 and e_6 are ordered by m_1 and m_2 respectively.

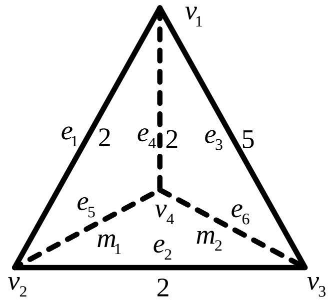


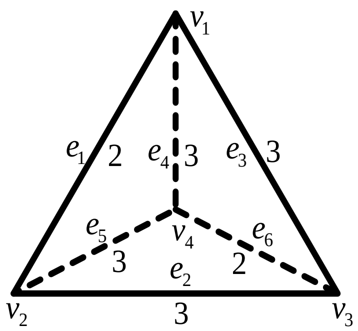
Figure 3.34

Then the possibilities for $m_1 = 2,3,4,5$. For $m_1 = 2,3,4$, the order of v_2 becomes $(2,2,2),(2,2,3),(2,2,4)$ respectively, which are not taken by assumption. For $m_1 = 5$, it falls in case 2. Therefore, there is no tetrahedron of this type.

Theorem 3.19: *In a coxeter Andreev's tetrahedron T. If T has no edge of order $n \geq 6$, then there are exactly 3 such T upto symmetry with at least one vertex is of order $(2,3,3)$ and no vertex is of order of the forms* $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$.

Proof: Let T be a coxeter Andreev's tetrahedron. Also let T has no edge of order $n \geq 6$, at least one vertex is of order $(2,3,3)$ and no vertex is of order $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$. By corollary 3.12, the number of same order vertices can be either 2 or 4. Therefore, there will be three cases: Case 1: all (four) the vertices are of order $(2,3,3)$, Case 2: two vertices are of order $(2,3,3)$ and Case 3: one vertex is of order $(2,3,3)$.

Case 1: All the vertices of T are of order $(2,3,3)$.
In this case, there are exactly one T of this type upto symmetry.



$T_{0n}\text{-}25 = [2,3,3,3,3,2]$

Figure 3.35

Case 2: Two vertices are of order $(2,3,3)$.
If the two vertices of order $(2,3,3)$ are with adjacent edges of order 2, then the order of the vertex at which the two edges of order 2 are adjacent becomes $(2,2,n \geq 2)$. This is not taken by assumption. Therefore, the two

vertices of order $(2,3,3)$ cannot be with adjacent edges of order 2 and hence, we can have the two vertices of order $(2,3,3)$ with either disjoint edges of order 2 or common edge of order 2

Case 2.1: Two vertices of order $(2,3,3)$ with disjoint edges of order 2.

Suppose the vertices v_1 and v_3 are of order $(2,3,3)$ with disjoint edges e_1 and e_6 of order 2. Then, the possibilities of orders for e_5 are $m = 2,3,4,5$.

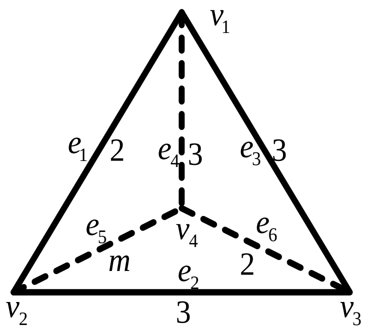
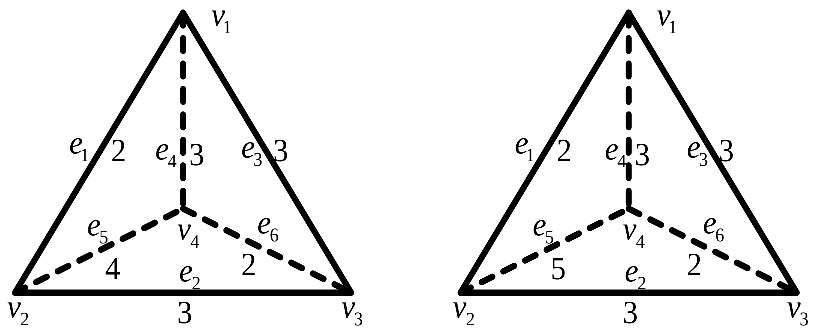


Figure 3.36

For $m = 2$, the order of v_2 becomes $(2,2,2)$ which is not taken by assumption. For $m = 3$, it falls in case 1. For $m = 4,5$, there are exactly 2 T of this type upto symmetry.



$T_{0n}\text{-}26 = [2,3,3,3,4,2]$ $T_{0n}\text{-}27 = [2,3,3,3,5,2]$

Figure 3.37

Case 2.2: Two vertices of order $(2,3,3)$ with common edge of order 2.

Suppose the vertices v_1 and v_2 are of order $(2,3,3)$ with common edge e_1 of order 2.

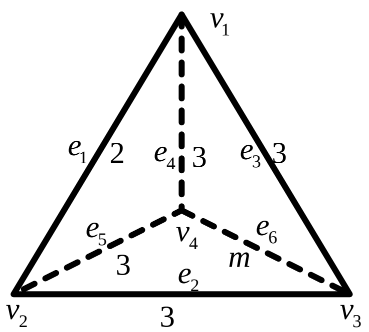


Figure 3.38

Then the possibilities of order for e_6 are $m = 2,3,4,5$. For $m = 2$, it falls in case 1. For $m = 3,4,5$, the order of v_3 becomes $(3,3,3),(3,3,4),(3,3,5)$ respectively, which are not possible by corollary 3.3. Hence there is no tetrahedron of this type.

Case 3: One vertex v_1 is of order $(2,3,3)$.
By corollary 3.6, the number of edges of order 2 at one vertex is at least 1 and at most 3. Also, the edges of order 2 must be disjoint as we do not have the vertices of the forms: $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$. Therefore at one vertex, there are exactly one edge of order 2. If v_1 is of order $(2,3,3)$ with e_1 is of order 2, then e_6 must be of order 2. Suppose, the order of the edges e_2 and e_5 are m_1 and m_2 respectively.

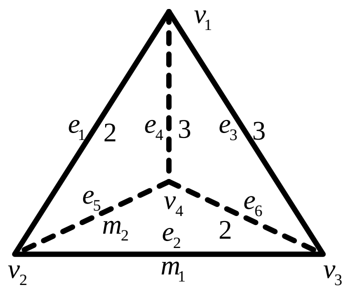


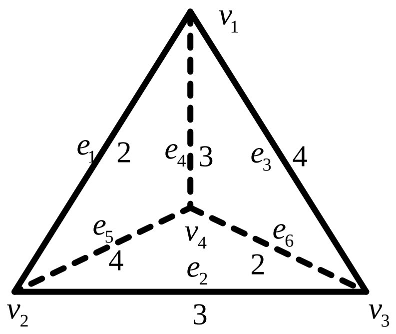
Figure 3.39

To avoid symmetry, assume $m_1 \le m_2$. Therefore $m_1 = 3$ and $m_2 = 4,5$ upto symmetry. But these falls in case 2. Hence, there is no such T of this type.

Theorem 3.20: *In a coxeter Andreev's tetrahedron T. If T has no edge of order $n \ge 6$, then there are exactly 2 such T upto symmetry with at least one vertex is of order $(2,3,4)$ and no vertex is of order of the forms* $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$,
$(2,3,3)$.

Proof: Let T be a coxeter Andreev's tetrahedron. Also let T has no edge of order $n \ge 6$, at least one vertex is of order $(2,3,4)$ and no vertex is of order $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$,
$(2,3,3)$. By corollary 3.12, the number of same order vertices can be either 2 or 4. Therefore, there will be three cases: Case 1: all (four) the vertices are of order $(2,3,4)$, Case 2: two vertices are of order $(2,3,4)$ and Case 3: one vertex is of order $(2,3,4)$.

Case 1: All the vertices of T are of order $(2,3,4)$.
In this case, we have exactly 1 T of this type upto symmetry.



$T_{0n}\text{-}28 = [2,3,4,3,4,2]$

Figure 3.40

Case 2: Two vertices are of order $(2,3,4)$.

If the two vertices of order $(2,3,4)$ are with adjacent edges of order 2, then the order of the vertex at which the two edges of order 2 are adjacent becomes $(2,2,n \geq 2)$. This is not taken by assumption. Therefore, the two vertices of order $(2,3,4)$ cannot be with adjacent edges of order 2 and hence, we can have the two vertices of order $(2,3,4)$ with either disjoint edges of order 2 or common edge of order 2.

Case 2.1: Two vertices of order $(2,3,4)$ with disjoint edges of order 2.

Suppose the vertices v_1 and v_3 are of order $(2,3,4)$ with disjoint edges e_1 and e_6 of order 2.

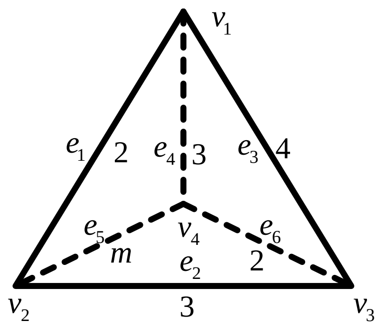
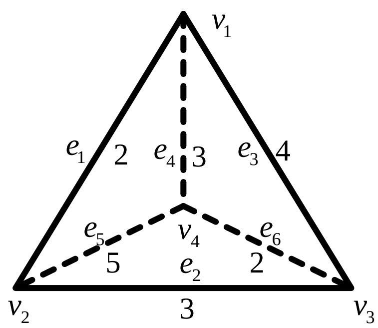


Figure 3.41

Then the possibilities of orders for e_5 are $m = 2,3,4,5$. For $m = 2,3$, the order of v_2 becomes $(2,2,3),(2,3,3)$ respectively, which are not taken by assumption. For $m = 4$, it falls in case 1. For $m = 5$, there are exactly 1 such T of this type upto symmetry.



$T_{0n}\text{-}29 = [2,3,4,3,5,2]$

Figure 3.42

Case 2.2: Two vertices of order $(2,3,4)$ with common edge of order 2.

Suppose the vertices v_1 and v_2 are of order $(2,3,4)$ with common edge e_1 of order 2.

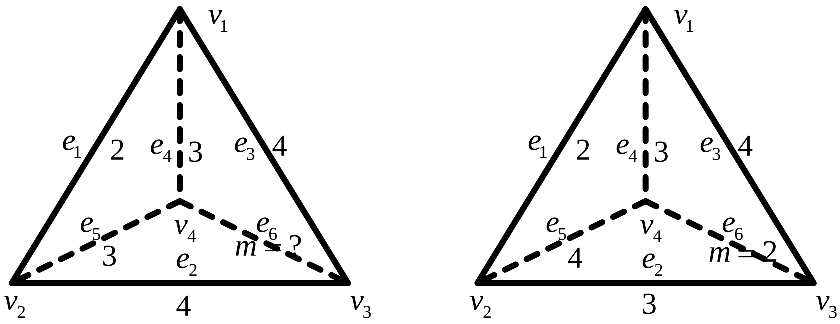


Figure 3.43

In the first figure, we do not have any choice for m. In the second figure, we have $m = 2$ and it falls in case 1.

Case 3: One vertex v_1 is of order $(2,3,4)$.

By corollary 3.6, the number of edges of order 2 at one vertex is at least 1 and at most 3. Also, the edges of order 2 must be disjoint as we do not have the vertices of the forms: $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$. Therefore at one vertex, there are exactly one edge of order 2. If v_1 is of order $(2,3,4)$ with e_1 is of order 2, then e_6 must be of order 2. Suppose, the order of the edges e_2 and e_5 are m_1 and m_2 respectively.

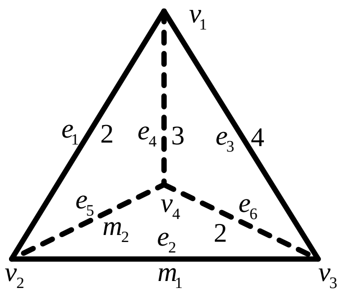


Figure 3.44

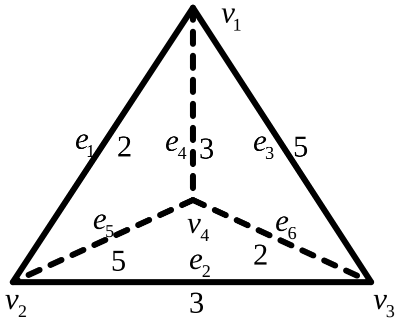
To avoid symmetry, assume $m_1 \le m_2$. Therefore, $m_1 = 3$ and $m_2 = 3,4,5$ upto symmetry. For $m_1 = 3$ and $m_2 = 3$, the order of v_2 becomes $(2,3,3)$, which cannot be taken by assumption and $m_1 = 3$, $m_2 = 4$ as well as $m_1 = 3$, $m_2 = 5$ lead to case 2. Hence, there is no such T of this type.

Theorem 3.21: *In a coxeter Andreev's tetrahedron T. If T has no edge of order $n \ge 6$, then there are exactly 1 such T upto symmetry with at least one vertex is of order $(2,3,5)$ and no vertex is of order of the forms* $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$, $(2,3,3),(2,3,4)$.

Proof: Let T be a coxeter Andreev's tetrahedron T. Also let T has no edge of order $n \ge 6$, at least one vertex is of order $(2,3,5)$ and no vertex is of order $(2,2,2),(2,2,3),(2,2,4)$,

$(2,2,5),(2,3,3),(2,3,4)$. By corollary 3.12, the number of same order vertices can be either 2 or 4. Therefore, there will be three cases: Case 1: all (four) the vertices are of order $(2,3,5)$, Case 2: two vertices are of order $(2,3,5)$ and Case 3: one vertex is of order $(2,3,5)$.

Case 1: All the vertices of T are of order $(2,3,5)$.
In this case, we have exactly 1 T upto symmetry.



$\mathrm{T}_{0n}\text{-}30=[2,3,5,3,5,2]$

Figure 3.45

Case 2: Two vertices are of order $(2,3,5)$.
If the two vertices of order $(2,3,5)$ are with adjacent edges of order 2, then the order of the vertex at which the two edges of order 2 are adjacent becomes $(2,2,n\geq 2)$. This cannot be taken by assumption. Therefore, the two vertices of order $(2,3,5)$ cannot be with adjacent edges of order 2 and hence, we can have the two vertices of order $(2,3,5)$ with either disjoint edges of order 2 or common edge of order 2

Case 2.1: Two vertices of order $(2,3,5)$ with disjoint edges of order 2.
Suppose the vertices v_1 and v_3 are of order $(2,3,5)$ with disjoint edges e_1 and e_6 of order 2.

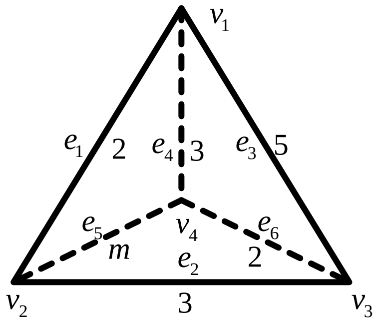


Figure 3.46

Then the possibilities of orders for e_5 are $m=2,3,4,5$. For $m=2,3,4$, the order of v_2 becomes $(2,2,3),(2,3,3),(2,3,4)$ respectively, which cannot be taken by assumption. For $m=5$, it falls in case 1. Hence, there is no such T of this type.

Case 2.2: Two vertices of order $(2,3,5)$ with common edge of order 2.

Suppose the vertices v_1 and v_2 are of order $(2,3,5)$ with common edge e_1 of order 2 upto symmetry.

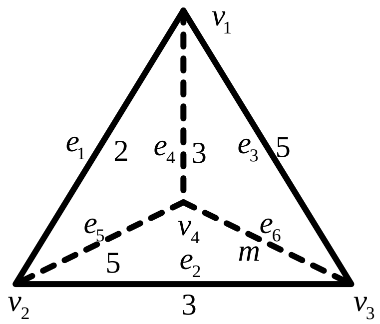


Figure 3.47

Then the only possibility of order for e_6 is $m = 2$ and this falls in case 1. Hence, there is no such T of this type.

Case 3: One vertex v_1 is of order $(2,3,5)$.

By corollary 3.6, the number of edges of order 2 at one vertex is at least 1 and at most 3. Also, the edges of order 2 must be disjoint as we do not have the vertices of the forms: $(2,2,2),(2,2,3),(2,2,4),(2,2,5)$. Therefore at one vertex, there are exactly one edge of order 2. If v_1 is of order $(2,3,5)$ with e_1 is of order 2, then e_6 must be of order 2. Suppose, the order of the edges e_2 and e_5 are m_1 and m_2 respectively.

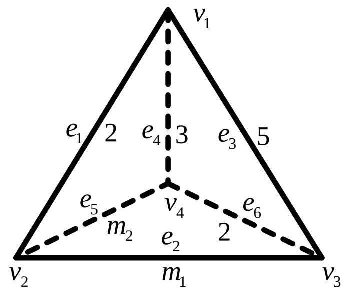


Figure 3.48

To avoid symmetry, assume $m_1 \le m_2$. Therefore, $m_1 = 3$ and $m_2 = 3,4,5$ upto symmetry. For $m_1 = 3$, $m_2 = 3$ and for $m_1 = 3$, $m_2 = 4$, the order of v_2 becomes $(2,3,3)$ and $(2,3,4)$ respectively, which cannot be taken by assumption. For $m_1 = 3$, $m_2 = 5$, it falls in case 1. Hence, there is no such T of this type.

From theorem 3.15 to theorem 3.21, the total number of coxeter Andreev's tetrahedrons with no edge of order $n \ge 6$ upto symmetry is $10+8+4+2+3+2+1=30$.

IV. Conclusions

In this article, it has been found that there are exactly one, four and thirty coxeter Andreev's tetrahedrons having respectively two edges of order $n \ge 6$, one edge of order $n \ge 6$ and no edge of order $n \ge 6$, $n \in N$ upto symmetry. These tetrahedrons may not be realized in Hyperbolic space. We can extend our research to find the coxeter Andreev's tetrahedrons which can be realized in Hyperbolic space. This research can also be extended to other compact as well as non-compact polytopes in spaces of different dimensions.

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