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Some Properties of Bipolar Fuzzy Cosets of a Bipolar Fuzzy and Bipolar Anti Fuzzy HX Subgroup

R. Muthuraj[1], M.Sridharan[2]
[1] PG & Research Department of Mathematics , H.H.The Rajah's College,Pudukkottai – 622 001, Tamilnadu, India.
[2]Department of Mathematics, PSNA College of Engineering and Technology, Dindigul-624 622, Tamilnadu , India.

Abstract : *In this paper, we define the concept of a bipolar fuzzy cosets of a bipolar fuzzy HX subgroup, bipolar anti fuzzy HX subgroup, bipolar fuzzy middle coset of a bipolar fuzzy HX subgroup and bipolar anti fuzzy HX subgroup, conjugate bipolar fuzzy HX subgroup, conjugate bipolar anti fuzzy HX subgroup and discussed some of its properties with the examples. Further we define the level subsets and lower level subsets of bipolar fuzzy cosets of a bipolar fuzzy and bipolar anti fuzzy HX subgroup and discussed some of its properties.*
Keywords: *bipolar **fuzzy** HX subgroup, bipolar anti fuzzy HX subgroup, bipolar fuzzy cosets of a bipolar fuzzy HX subgroup, bipolar fuzzy middle cosets, conjugate bipolar fuzzy HX subgroup .*

I. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh [8]. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld [10] gave the idea of fuzzy subgroups. Li Hongxing [3] introduce the concept of HX group and the authors Luo Chengzhong , Mi Honghai , Li Hongxing [4] introduce the concept of fuzzy HX group. The author W.R.Zhang [12] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets in 1994. In fuzzy sets the membership degree of elements range over the interval [0,1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0, 1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set .In case of Bipolar-valued fuzzy sets membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree (0,1] indicates that elements somewhat satisfy the property and the membership degree [−1,0) indicates that elements somewhat satisfy the implicit counter-property. R.Muthuraj et.al [6],[7] introduce the concept of bipolar fuzzy HX subgroup and bipolar anti fuzzy HX subgroup. W. B. Vasantha kandasamy [11] introduced the concept of fuzzy cosets and fuzzy middle cosets of a group. In this paper we define the concept of bipolar fuzzy cosets of a bipolar fuzzy HX subgroup and bipolar anti fuzzy HX subgroup , bipolar fuzzy middle cosets of a bipolar fuzzy HX subgroup and bipolar anti fuzzy HX subgroup and discussed some of their related properties.

II. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper G = (G , *) is a finite group, e is the identity element of G, and xy we mean x * y.

2.1 Definition [4]

Let G be a finite group. In $2^G - \{\phi\}$, a non-empty set $\vartheta \subset 2^G - \{\phi\}$ is called a HX group of G, if ϑ is a group with respect to the algebraic operation defined by AB = { ab /a ∈ A and b ∈ B}, which its unit element is denoted by E.

2.2 Definition [9]

Let X be any non-empty set. A fuzzy subset μ of X is a function $\mu : X \rightarrow [0,1]$.

2.3 Definition [6],[7]

Let ϑ be a non-empty set . A bipolar-valued fuzzy set or bipolar fuzzy set λ_μ in ϑ is an object having the form $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$ where $\lambda_\mu^+ : \vartheta \rightarrow [0,1]$ and $\lambda_\mu^- : \vartheta \rightarrow [-1,0]$ are mappings. The positive membership degree $\lambda_\mu^+(A)$ denotes the satisfaction degree of an element A to the property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$ and the negative membership degree $\lambda_\mu^-(A)$ denotes the satisfaction degree of an element A to some implicit counter property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$. If $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) = 0$, it is the situation that A is regarded as having only positive satisfaction for $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$. If $\lambda_\mu^+(A) = 0$ and $\lambda_\mu^-(A) \neq 0$, it is the situation that A does not satisfy the property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$, but somewhat satisfies the counter property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$. It is possible for an element A to be such that $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) \neq 0$ when the membership function of property overlaps that its counter property over some portion of ϑ. For the sake of simplicity, we shall use the symbol $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ for the bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$.

2.4 Definition [6],[7]

Let μ be a bipolar fuzzy subset defined on G . Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group of G. A bipolar fuzzy set λ_μ defined on ϑ is said to be a bipolar fuzzy subgroup induced by μ on ϑ or a bipolar fuzzy HX subgroup of ϑ if for A , B $\in \vartheta$,

i. $\lambda_\mu^+(AB) \geq \min\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$
ii. $\lambda_\mu^-(AB) \leq \max\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$
iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$.

where $\lambda_\mu^+(A) = \max\{\mu^+(x) / \text{for all } x \in A \subseteq G\}$ and
$\lambda_\mu^-(A) = \min\{\mu^-(x) / \text{for all } x \in A \subseteq G\}$.

2.5 Definition [6],[7]

Let μ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group of G. A bipolar fuzzy set λ_μ defined on ϑ is said to be a bipolar anti fuzzy subgroup induced by μ on ϑ or a bipolar anti fuzzy HX subgroup of ϑ . if for A, B $\in \vartheta$,

i. $\lambda_\mu^+(AB) \leq \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$
ii. $\lambda_\mu^-(AB) \geq \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$
iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$.

where $\lambda_\mu^+(A) = \min\{\mu^+(x) / \text{for all } x \in A \subseteq G\}$ and
$\lambda_\mu^-(A) = \max\{\mu^-(x) / \text{for all } x \in A \subseteq G\}$

2.6 Definition

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of a group G and let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of a HX group ϑ with $\lambda_\mu^+(A) = \max\{\mu^+(x) / \text{for all } x \in A \subseteq G\}$ and $\lambda_\mu^-(A) = \min\{\mu^-(x) / \text{for all } x \in A \subseteq G\}$. For any $A \in \vartheta$, the bipolar fuzzy coset $A\lambda_\mu = (A\lambda_\mu^+, A\lambda_\mu^-)$ of a bipolar fuzzy HX group λ_μ of ϑ determined by A is defined by

i. $(A\lambda_\mu^+)(X) = \lambda_\mu^+(A^{-1}X)$
ii. $(A\lambda_\mu^-)(X) = \lambda_\mu^-(A^{-1}X)$ for every $X \in \vartheta$.

2.7 Definition

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of a group G and let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of a HX group ϑ with $\lambda_\mu^+(A) = \min\{\mu^+(x) / \text{for all } x \in A \subseteq G\}$ and $\lambda_\mu^-(A) = \max\{\mu^-(x) / \text{for all } x \in A \subseteq G\}$. For any $A \in \vartheta$, the bipolar fuzzy coset $A\lambda_\mu = (A\lambda_\mu^+, A\lambda_\mu^-)$ of a bipolar anti fuzzy HX group λ_μ of ϑ determined by A is defined by

i. $(A\lambda_\mu^+)(X) = \lambda_\mu^+(A^{-1}X)$
ii. $(A\lambda_\mu^-)(X) = \lambda_\mu^-(A^{-1}X)$ for every $X \in \vartheta$.

2.8 Example

Let G be the Klein's 4 group. Then G = {e, a, b, ab} where $a^2 = e = b^2$, ab = ba and e the identity element of G.

Define the bipolar fuzzy subset $\mu = (\mu^+, \mu^-)$ on G as

$\mu^+(x) = 0.7$ if $x = e$, $\mu^+(x) = 0.4$ if $x = a$, $\mu^+(x) = 0.3$ if $x = b, ab$

$\mu^-(x) = -0.7$ if $x = e$, $\mu^-(x) = -0.4$ if $x = a$, $\mu^-(x) = -0.3$ if $x = b, ab$

Let $\vartheta = \{E,A\} = \{\{e,a\}, \{b , ab\}\}$ be a HX group on G.

and define $\lambda_\mu^+ : \vartheta \to [0,1]$, $\lambda_\mu^- : \vartheta \to [-1,0]$ as $\lambda_\mu^+(A) = \max\{\mu^+(x) / \text{for all } x \in A \subseteq G\}$ and
$\lambda_\mu^-(A) = \min\{\mu^-(x) / \text{for all } x \in A \subseteq G\}$

$\lambda_\mu^+ (X) = \begin{cases} 0.7 & \text{if } X = E \\ 0.3 & \text{if } X = A \end{cases}$ $\lambda_\mu^- (X) = \begin{cases} -0.7 & \text{if } X = E \\ -0.3 & \text{if } X = A \end{cases}$

Clearly, λ_μ is a bipolar fuzzy HX subgroup of a HX group ϑ. Now, we compute the bipolar fuzzy cosets of λ_μ.

$(E\lambda_\mu^+) (X) = \begin{cases} 0.7 & \text{if } X = E \\ 0.3 & \text{if } X = A \end{cases}$ $(E\lambda_\mu^-) (X) = \begin{cases} -0.7 & \text{if } X = E \\ -0.3 & \text{if } X = A \end{cases}$

and

$(A\lambda_\mu^+) (X) = \begin{cases} 0.3 & \text{if } X = E \\ 0.7 & \text{if } X = A \end{cases}$ $(A\lambda_\mu^-) (X) = \begin{cases} -0.3 & \text{if } X = E \\ -0.7 & \text{if } X = A, B \end{cases}$

Remark:

i. If A = E, then bipolar fuzzy coset $A\lambda_\mu = \lambda_\mu$
ii. If λ_μ is a bipolar fuzzy HX subgroup of a group G, and A = E then bipolar fuzzy coset $(A\lambda_\mu)$ is also a bipolar fuzzy HX subgroup of G.
iii. If λ_μ is a bipolar anti fuzzy HX subgroup of a group G, and A = E then bipolar fuzzy coset $(A\lambda_\mu)$ is also a bipolar anti fuzzy HX subgroup of G.

2.9 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of a HX group ϑ then the bipolar fuzzy coset $(A\lambda_\mu) = (A\lambda_\mu^+, A\lambda_\mu^-)$ is a bipolar fuzzy HX subgroup of ϑ if

i. $\min\{ \lambda_\mu^+(A^{-1}Y), \lambda_\mu^+(A)\} = \lambda_\mu^+(A^{-1}Y)$
ii. $\max\{ \lambda_\mu^-(A^{-1}Y), \lambda_\mu^-(A)\} = \lambda_\mu^-(A^{-1}Y)$, for every $Y \in \vartheta$.

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of ϑ. For every $X, Y \in \vartheta$, we have,

i.
$$\begin{aligned}(A\lambda_\mu^+)(XY^{-1}) &= \lambda_\mu^+(A^{-1}XY^{-1}) \\ &= \lambda_\mu^+(A^{-1}XY^{-1}AA^{-1}) \\ &\geq \min\{\lambda_\mu^+(A^{-1}X), \lambda_\mu^+(Y^{-1}AA^{-1})\} \\ &\geq \min\{\lambda_\mu^+(A^{-1}X), \min\{\lambda_\mu^+(Y^{-1}A), \lambda_\mu^+(A^{-1})\}\} \\ &= \min\{\lambda_\mu^+(A^{-1}X), \min\{\lambda_\mu^+((Y^{-1}A)^{-1}), \lambda_\mu^+(A)\}\} \\ &= \min\{\lambda_\mu^+(A^{-1}X), \min\{\lambda_\mu^+(A^{-1}Y), \lambda_\mu^+(A)\}\} \\ &= \min\{\lambda_\mu^+(A^{-1}X), \lambda_\mu^+(A^{-1}Y)\} \\ &= \min\{(A\lambda_\mu^+)(X), (A\lambda_\mu^+)(Y)\}\end{aligned}$$

Therefore, $(A\lambda_\mu^+)(XY^{-1}) \geq \min\{A\lambda_\mu^+(X), A\lambda_\mu^+(Y)\}$

ii.
$$\begin{aligned}(A\lambda_\mu^-)(XY^{-1}) &= \lambda_\mu^-(A^{-1}XY^{-1}) \\ &= \lambda_\mu^-(A^{-1}XY^{-1}AA^{-1}) \\ &\leq \max\{\lambda_\mu^-(A^{-1}X), \lambda_\mu^-(Y^{-1}AA^{-1})\} \\ &\leq \max\{\lambda_\mu^-(A^{-1}X), \max\{\lambda_\mu^-(Y^{-1}A), \lambda_\mu^-(A^{-1})\}\} \\ &= \max\{\lambda_\mu^-(A^{-1}X), \max\{\lambda_\mu^-((Y^{-1}A)^{-1}), \lambda_\mu^-(A)\}\} \\ &= \max\{\lambda_\mu^-(A^{-1}X), \max\{\lambda_\mu^-(A^{-1}Y), \lambda_\mu^-(A)\}\} \\ &= \max\{\lambda_\mu^-(A^{-1}X), \lambda_\mu^-((A^{-1}Y)\} \\ &= \max\{(A\lambda_\mu^-)(X), (A\lambda_\mu^-)(Y)\}\end{aligned}$$

Therefore , $(A\lambda_\mu^-)(XY^{-1}) \leq \max\{(A\lambda_\mu^-)(X), (A\lambda_\mu^-)(Y)\}$

Hence, $A\lambda_\mu = (A\lambda_\mu^+, A\lambda_\mu^-)$ is a bipolar fuzzy HX subgroup of ϑ.

2.10 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of a HX group ϑ then the bipolar fuzzy coset $(A\lambda_\mu) = (A\lambda_\mu^+, A\lambda_\mu^-)$ is a bipolar anti fuzzy HX subgroup of ϑ if

i. $\max\{ \lambda_\mu^+(A^{-1}Y), \lambda_\mu^+(A)\} = \lambda_\mu^+(A^{-1}Y)$
ii. $\min\{ \lambda_\mu^-(A^{-1}Y), \lambda_\mu^-(A)\} = \lambda_\mu^-(A^{-1}Y)$, for every $Y \in \vartheta$.

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of a HX group ϑ. For every $X, Y \in \vartheta$, we have,

i.
$$\begin{aligned}(A\lambda_\mu^+)(XY^{-1}) &= \lambda_\mu^+(A^{-1}XY^{-1}) \\ &= \lambda_\mu^+(A^{-1}XY^{-1}AA^{-1}) \\ &\leq \max\{\lambda_\mu^+(A^{-1}X), \lambda_\mu^+(Y^{-1}AA^{-1})\} \\ &\leq \max\{\lambda_\mu^+(A^{-1}X), \max\{\lambda_\mu^+(Y^{-1}A), \lambda_\mu^+(A^{-1})\}\} \\ &= \max\{\lambda_\mu^+(A^{-1}X), \max\{\lambda_\mu^+((Y^{-1}A)^{-1}), \lambda_\mu^+(A)\}\}\end{aligned}$$

$= \max \{ \lambda_\mu^+ (A^{-1} X), \max\{ \lambda_\mu^+ ((A^{-1}Y), \lambda_\mu^+ (A)\}\}$
$= \max \{ \lambda_\mu^+ (A^{-1} X), \lambda_\mu^+ ((A^{-1}Y)\}$
$= \max \{(A\lambda_\mu^+)(X), (A\lambda_\mu^+)(Y)\}$

Therefore, $(A\lambda_\mu^+) (XY^{-1}) \leq \max \{(A\lambda_\mu^+)(X), (A\lambda_\mu^+)(Y)\}$

ii. $(A\lambda_\mu^-) (XY^{-1}) = \lambda_\mu^- (A^{-1} XY^{-1})$
$= \lambda_\mu^- (A^{-1} XY^{-1} AA^{-1})$
$\geq \min \{ \lambda_\mu^- (A^{-1} X) , \lambda_\mu^- (Y^{-1} AA^{-1})\}$
$\geq \min \{ \lambda_\mu^- (A^{-1} X), \min \{ \lambda_\mu^- (Y^{-1}A), \lambda_\mu^- (A^{-1})\}$
$= \min \{ \lambda_\mu^- (A^{-1} X), \min\{ \lambda_\mu^- ((Y^{-1}A)^{-1}), \lambda_\mu^- (A)\}\}$
$= \min \{ \lambda_\mu^- (A^{-1} X), \min\{ \lambda_\mu^- ((A^{-1}Y), \lambda_\mu^- (A)\}\}$
$= \min \{ \lambda_\mu^- (A^{-1} X), \lambda_\mu^- ((A^{-1}Y)\}$
$= \min \{ (A\lambda_\mu^-)(X), (A\lambda_\mu^-)(Y)\}$

Therefore, $(A\lambda_\mu^-) (XY^{-1}) \geq \min \{ (A\lambda_\mu^-)(X), (A\lambda_\mu^-)(Y)\}$

Hence, $A\lambda_\mu = (A\lambda_\mu^+, A\lambda_\mu^-)$ is a bipolar anti fuzzy HX subgroup of ϑ .

2.11 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+ , \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of a HX group ϑ then $X\lambda_\mu = Y\lambda_\mu$, for $X,Y \in \vartheta$ if and only if

i. $\lambda_\mu^+(X^{-1}Y) = \lambda_\mu^+ (Y^{-1}X) = \lambda_\mu^+ (E)$
ii. $\lambda_\mu^-(X^{-1}Y) = \lambda_\mu^- (Y^{-1}X) = \lambda_\mu^- (E)$

Proof : Let $\lambda_\mu = (\lambda_\mu^+ , \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of a HX group ϑ.

Let $X\lambda_\mu = Y\lambda_\mu$, for $X,Y \in \vartheta$,that is i. $X\lambda_\mu^+ = Y\lambda_\mu^+$ ii. $X\lambda_\mu^- = Y\lambda_\mu^-$

Then, $(X\lambda_\mu^+)(X) = (Y\lambda_\mu^+)(X)$, $(X\lambda_\mu^+)(Y) = (Y\lambda_\mu^+)(Y)$ and
$(X\lambda_\mu^-)(X) = (Y\lambda_\mu^-)(X)$, $(X\lambda_\mu^-)(Y) = (Y\lambda_\mu^-)(Y)$ which implies that,
$\lambda_\mu^+ (X^{-1}X) = \lambda_\mu^+ (Y^{-1}X)$, $\lambda_\mu^+ (X^{-1}Y) = \lambda_\mu^+ (Y^{-1}Y)$ and
$\lambda_\mu^- (X^{-1}X) = \lambda_\mu^- (Y^{-1}X)$, $\lambda_\mu^- (X^{-1}Y) = \lambda_\mu^- (Y^{-1}Y)$ (By definition)
$\Rightarrow \lambda_\mu^+ (E) = \lambda_\mu^+ (Y^{-1}X)$, $\lambda_\mu^+ (X^{-1}Y) = \lambda_\mu^+ (E)$ and
$\lambda_\mu^- (E) = \lambda_\mu^- (Y^{-1}X)$, $\lambda_\mu^- (X^{-1}Y) = \lambda_\mu^- (E)$.

Hence , $\lambda_\mu^+ (X^{-1}Y) = \lambda_\mu^+ (Y^{-1}X) = \lambda_\mu^+ (E)$ and $\lambda_\mu^- (X^{-1}Y) = \lambda_\mu^- (Y^{-1}X) = \lambda_\mu^- (E)$.

Conversely,

For $X,Y\in \vartheta$. Let $\lambda_\mu^+ (X^{-1}Y) = \lambda_\mu^+ (Y^{-1}X) = \lambda_\mu^+ (E)$ and $\lambda_\mu^- (X^{-1}Y) = \lambda_\mu^- (Y^{-1}X) = \lambda_\mu^- (E)$,

For every $U\in \vartheta$, we have,

i. $(X\lambda_\mu^+)(U) = \lambda_\mu^+ (X^{-1}U)$
$= \lambda_\mu^+ (X^{-1} YY^{-1}U)$
$\geq \min\{ \lambda_\mu^+ (X^{-1}Y), \lambda_\mu^+ (Y^{-1}U) \}$
$= \min \{ \lambda_\mu^+ (E), \lambda_\mu^+ (Y^{-1}U) \}$
$= \lambda_\mu^+ (Y^{-1}U)$
$= (Y\lambda_\mu^+)(U)$.

Therefore, $(X\lambda_\mu^+)(U) \geq (Y\lambda_\mu^+)(U)$.
similarly, $(Y\lambda_\mu^+)(U) \geq (X\lambda_\mu^+)(U)$.
Hence, $(X\lambda_\mu^+)(U) = (Y\lambda_\mu^+)(U)$.

ii. $(X\lambda_\mu^-)(U) = \lambda_\mu^- (X^{-1}U)$
$= \lambda_\mu^- (X^{-1}YY^{-1}U)$
$\leq \max \{ \lambda_\mu^- (X^{-1}Y), \lambda_\mu^- (Y^{-1}U) \}$
$= \max \{ \lambda_\mu^- (E), \lambda_\mu^- (Y^{-1}U) \}$
$= \lambda_\mu^- (Y^{-1}U)$
$= Y\mu^- (U)$.

Therefore, $(X\lambda_\mu^-)(U) \leq (Y\lambda_\mu^-)(U)$.
similarly, $(Y\lambda_\mu^-)(U) \leq (X\lambda_\mu^-)(U)$.
Hence, $(X\lambda_\mu^-)(U) = (Y\lambda_\mu^-)(U)$.
Hence, $X\lambda_\mu = Y\lambda_\mu$, for $X,Y \in \vartheta$.

2.12 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of a HX group ϑ then $X\lambda_\mu = Y\lambda_\mu$, for $X,Y \in \vartheta$ if and only if

i. $\lambda_\mu^+(X^{-1}Y) = \lambda_\mu^+(Y^{-1}X) = \lambda_\mu^+(E)$

ii. $\lambda_\mu^-(X^{-1}Y) = \lambda_\mu^-(Y^{-1}X) = \lambda_\mu^-(E)$

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of a HX group ϑ.

Let $X\lambda_\mu = Y\lambda_\mu$, for $x, y \in G$ that is i. $X\lambda_\mu^+ = Y\lambda_\mu^+$ ii. $X\lambda_\mu^- = Y\lambda_\mu^-$

Then, $(X\lambda_\mu^+)(X) = (Y\lambda_\mu^+)(X)$, $(X\lambda_\mu^+)(Y) = (Y\lambda_\mu^+)(Y)$ and

$(X\lambda_\mu^-)(X) = (Y\lambda_\mu^-)(X)$, $(X\lambda_\mu^-)(Y) = (Y\lambda_\mu^-)(Y)$ which implies that,

$\lambda_\mu^+(X^{-1}X) = \lambda_\mu^+(Y^{-1}X)$, $\lambda_\mu^+(X^{-1}Y) = \lambda_\mu^+(Y^{-1}Y)$ and

$\lambda_\mu^-(X^{-1}X) = \lambda_\mu^-(Y^{-1}X)$, $\lambda_\mu^-(X^{-1}Y) = \lambda_\mu^-(Y^{-1}Y)$ (By definition of bipolar fuzzy coset)

$\Rightarrow \lambda_\mu^+(E) = \lambda_\mu^+(Y^{-1}X)$, $\lambda_\mu^+(X^{-1}Y) = \lambda_\mu^+(E)$ and

$\lambda_\mu^-(E) = \lambda_\mu^-(Y^{-1}X)$, $\lambda_\mu^-(X^{-1}Y) = \lambda_\mu^-(E)$.

Hence , $\lambda_\mu^+(X^{-1}Y) = \lambda_\mu^+(Y^{-1}X) = \lambda_\mu^+(E)$ and $\lambda_\mu^-(X^{-1}Y) = \lambda_\mu^-(Y^{-1}X) = \lambda_\mu^-(E)$.

Conversely,

For $X,Y \in \vartheta$.Let $\lambda_\mu^+(X^{-1}Y) = \lambda_\mu^+(Y^{-1}X) = \lambda_\mu^+(E)$ and $\lambda_\mu^-(X^{-1}Y) = \lambda_\mu^-(Y^{-1}X) = \lambda_\mu^-(E)$,

For every $U \in \vartheta$, we have,

i. $(X\lambda_\mu^+)(U) = \lambda_\mu^+(X^{-1}U)$
$= \lambda_\mu^+(X^{-1}YY^{-1}U)$
$\leq \max\{\lambda_\mu^+(X^{-1}Y), \lambda_\mu^+(Y^{-1}U)\}$
$= \max\{\lambda_\mu^+(E), \lambda_\mu^+(Y^{-1}U)\}$
$= \lambda_\mu^+(Y^{-1}U)$
$= (Y\lambda_\mu^+)(U)$.

Therefore, $(X\lambda_\mu^+)(U) \leq (Y\lambda_\mu^+)(U)$.

similarly, $(Y\lambda_\mu^+)(U) \leq (X\lambda_\mu^+)(U)$.

Hence, $(X\lambda_\mu^+)(U) = (Y\lambda_\mu^+)(U)$.

ii. $(X\lambda_\mu^-)(U) = \lambda_\mu^-(X^{-1}U)$
$= \lambda_\mu^-(X^{-1}YY^{-1}U)$
$\geq \min\{\lambda_\mu^-(X^{-1}Y), \lambda_\mu^-(Y^{-1}U)\}$
$= \min\{\lambda_\mu^-(E), \lambda_\mu^-(Y^{-1}U)\}$
$= \lambda_\mu^-(Y^{-1}U)$
$= (Y\lambda_\mu^-)(U)$.

Therefore, $(X\lambda_\mu^-)(U) \geq (Y\lambda_\mu^-)(U)$.

similarly, $(Y\lambda_\mu^-)(U) \geq (X\lambda_\mu^-)(U)$.

Hence, $(X\lambda_\mu^-)(U) = (Y\lambda_\mu^-)(U)$.

Hence, $X\lambda_\mu = Y\lambda_\mu$, for $X, Y \in \vartheta$.

2.13 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of ϑ and $X\lambda_\mu = Y\lambda_\mu$, for $X,Y \in \vartheta$ then $\lambda_\mu(X) = \lambda_\mu(Y)$.

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of ϑ and $X\lambda_\mu = Y\lambda_\mu$, for $X,Y \in \vartheta$.

Now, i. $\lambda_\mu^+(X) = \lambda_\mu^+(YY^{-1}X)$
$\geq \min\{\lambda_\mu^+(Y), \lambda_\mu^+(Y^{-1}X)\}$
$= \min\{\lambda_\mu^+(Y), \lambda_\mu^+(E)\}$, by Theorem 2.3
$= \lambda_\mu^+(Y)$.

Therefore, $\lambda_\mu^+(X) \geq \lambda_\mu^+(Y)$.

Similarly, $\lambda_\mu^+(Y) \geq \lambda_\mu^+(X)$.

Hence, $\lambda_\mu^+(X) = \lambda_\mu^+(Y)$.

ii. $\lambda_\mu^-(X) = \lambda_\mu^-(YY^{-1}X)$
$\leq \max\{\lambda_\mu^-(Y), \lambda_\mu^-(Y^{-1}X)\}$
$= \max\{\lambda_\mu^-(Y), \lambda_\mu^-(E)\}$, by Theorem 2.3
$= \lambda_\mu^-(Y)$.

Therefore, $\lambda_\mu^-(X) \leq \lambda_\mu^-(Y)$.

Similarly, $\lambda_\mu^-(Y) \leq \lambda_\mu^-(X)$.

Hence, $\lambda_\mu^-(X) = \lambda_\mu^-(Y)$.

Hence, $\lambda_\mu(X) = \lambda_\mu(Y)$.

2.14 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy subgroup of ϑ and $X\lambda_\mu = Y\lambda_\mu$, for $X,Y \in \vartheta$ then $\lambda_\mu(X) = \lambda_\mu(Y)$.

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of ϑ and $X\lambda_\mu = Y\lambda_\mu$, for $X,Y \in \vartheta$.

Now, i. $\lambda_\mu^+(X) = \lambda_\mu^+(YY^{-1}X)$
$\leq \max\{\lambda_\mu^+(Y), \lambda_\mu^+(Y^{-1}X)\}$
$= \max\{\lambda_\mu^+(Y), \lambda_\mu^+(E)\}$, by Theorem 2.3
$= \lambda_\mu^+(Y)$.

Therefore, $\lambda_\mu^+(X) \leq \lambda_\mu^+(Y)$.
Similarly, $\lambda_\mu^+(Y) \leq \lambda_\mu^+(X)$.
Hence, $\lambda_\mu^+(X) = \lambda_\mu^+(Y)$.

ii. $\lambda_\mu^-(X) = \lambda_\mu^-(YY^{-1}X)$
$\geq \min\{\lambda_\mu^-(Y), \lambda_\mu^-(Y^{-1}X)\}$
$= \min\{\lambda_\mu^-(Y), \lambda_\mu^-(E)\}$, by Theorem 2.3
$= \lambda_\mu^-(Y)$.

Therefore, $\lambda_\mu^-(X) \geq \lambda_\mu^-(Y)$.
Similarly, $\lambda_\mu^-(Y) \geq \lambda_\mu^-(X)$.
Hence, $\lambda_\mu^-(X) = \lambda_\mu^-(Y)$.
Hence, $\lambda_\mu(X) = \lambda_\mu(Y)$.

2.15 Definition

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup (bipolar anti fuzzy HX subgroup) of a HX group ϑ. Then for any $A,B \in \vartheta$, a bipolar fuzzy middle coset $A\lambda_\mu B = (A\lambda_\mu^+ B, A\lambda_\mu^- B)$ of a bipolar fuzzy HX subgroup (bipolar anti fuzzy HX subgroup) λ_μ of ϑ determined by A and B is defined as

i. $(A\lambda_\mu^+ B)(X) = \lambda_\mu^+(A^{-1}XB^{-1})$
ii. $(A\lambda_\mu^- B)(X) = \lambda_\mu^-(A^{-1}XB^{-1})$.

2.16 Example

Let $G = \{Z_7 - \{0\}, \bullet_7\}$ be a group with respect to multiplication. Define the bipolar fuzzy subset $\mu = (\mu^+, \mu^-)$ on G as

$$\mu^+(x) = \begin{cases} 0.8 & \text{if } x = 1 \\ 0.5 & \text{if } x = 2,4 \\ 0.4 & \text{if } x = 3,5 \\ 0.3 & \text{if } x = 6 \end{cases} \qquad \mu^-(x) = \begin{cases} -0.8 & \text{if } x = 1 \\ -0.5 & \text{if } x = 2,4 \\ -0.4 & \text{if } x = 3,5 \\ -0.3 & \text{if } x = 6 \end{cases}$$

Clearly $\mu = (\mu^+, \mu^-)$ is a bipolar fuzzy subgroup of G. Let $\vartheta = \{E,A,B\}$, where $E = \{1,6\}$, $A = \{2,5\}$, $B = \{3,4\}$. Clearly $(\vartheta, \bullet_7)$ is a bipolar fuzzy HX subgroup.

Then the bipolar fuzzy middle coset $(A\lambda_\mu B) = (A\lambda_\mu^+ B, A\lambda_\mu^- B)$ is defined as

$$(A\lambda_\mu^+ B)(X) = \begin{cases} 0.8 & \text{if } X = E \\ 0.5 & \text{if } X = A, B \end{cases} \qquad (A\lambda_\mu^- B)(X) = \begin{cases} -0.8 & \text{if } X = E \\ -0.5 & \text{if } X = A, B \end{cases}$$

Clearly $(A\lambda_\mu B)$ is a bipolar fuzzy HX subgroup of ϑ since $B = A^{-1}$, But

$$(E\lambda_\mu^+ A)(X) = \begin{cases} 0.5 & \text{if } X = E \\ 0.8 & \text{if } X = A, B \end{cases} \qquad (A\lambda_\mu^- B)(X) = \begin{cases} -0.5 & \text{if } X = E \\ -0.8 & \text{if } X = A, B \end{cases}$$

The bipolar fuzzy middle coset $(E\lambda_\mu A)$ is not a bipolar fuzzy HX subgroup of ϑ.

Remark:

i. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of ϑ, then bipolar fuzzy middle coset $(A\lambda_\mu B) = (A\lambda_\mu^+ B, A\lambda_\mu^- B)$ is also a bipolar fuzzy HX subgroup of ϑ if $B = A^{-1}$.
ii. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of ϑ, then bipolar fuzzy middle coset

$(A\lambda_\mu B) = (A\lambda_\mu^+ B, A\lambda_\mu^- B)$ is also a bipolar anti fuzzy HX subgroup of ϑ if $B = A^{-1}$.

2.17 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of ϑ then the bipolar middle coset $(A\lambda_\mu A^{-1}) = (A\lambda_\mu^+ A^{-1}, A\lambda_\mu^- A^{-1})$ is a bipolar fuzzy HX subgroup of ϑ.

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of ϑ and Let for some $A \in \vartheta$, for every $X, Y \in \vartheta$ then

i. $(A\lambda_\mu^+ A^{-1})(XY^{-1})$ $= \lambda_\mu^+ (A^{-1}XY^{-1}A)$, by the definition
$= \lambda_\mu^+ (A^{-1}X AA^{-1}Y^{-1}A)$
$= \lambda_\mu^+ ((A^{-1}X A)(A^{-1}Y^{-1}A))$
$= \lambda_\mu^+ ((A^{-1}XA)(A^{-1}Y^{-1}A)^{-1})$
$\geq \min \{ \lambda_\mu^+ (A^{-1}X A), \lambda_\mu^+ (A^{-1}Y^{-1}A)^{-1}\}$
$\geq \min \{ \lambda_\mu^+ (A^{-1}XA), \lambda_\mu^+ (A^{-1}Y^{-1}A)\}$,
(since λ_μ is a bipolar fuzzy HX subgroup of ϑ)
$\geq \min \{(A\lambda_\mu^+ A^{-1})(X), (A\lambda_\mu^+ A^{-1})(Y)\}$.

Therefore, $(A\lambda_\mu^+ A^{-1})(XY^{-1}) \geq \min \{(A\lambda_\mu^+ A^{-1})(X), (A\lambda_\mu^+ A^{-1})(Y)\}$.

ii. $(A\lambda_\mu^- A^{-1})(XY^{-1})$ $= \lambda_\mu^- (A^{-1}XY^{-1}A)$, by the definition
$= \lambda_\mu^- (A^{-1}XAA^{-1}Y^{-1}A)$
$= \lambda_\mu^- ((A^{-1}XA)(A^{-1}Y^{-1}A))$
$= \lambda_\mu^- ((A^{-1}XA)(A^{-1}Y^{-1}A)^{-1})$
$\leq \max \{ \lambda_\mu^- (A^{-1}XA), \lambda_\mu^- (A^{-1}Y^{-1}A)^{-1}\}$
$\leq \max \{ \lambda_\mu^- (A^{-1}XA), \lambda_\mu^- (A^{-1}Y^{-1}A)\}$,
(since λ_μ is a bipolar fuzzy subgroup of ϑ)
$\leq \max \{(A\lambda_\mu^- A^{-1})(X), (A\lambda_\mu^- A^{-1})(Y)\}$

Therefore, $(A\lambda_\mu^- A^{-1})(XY^{-1}) \leq \max \{(A\lambda_\mu^- A^{-1})(X), (A\lambda_\mu^- A^{-1})(Y)\}$.

Hence, $(A\lambda_\mu A^{-1}) = (A\lambda_\mu^+ A^{-1}, A\lambda_\mu^- A^{-1})$ is a bipolar fuzzy HX subgroup of ϑ.

2.18 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of ϑ then the bipolar middle coset $(A\mu A^{-1}) = (A\mu^+ A^{-1}, A\mu^- A^{-1})$ is a bipolar anti fuzzy HX subgroup of ϑ.

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of ϑ and Let for some $A \in \vartheta$, for every $X, Y \in \vartheta$ then

i. $(A\lambda_\mu^+ A^{-1})(XY^{-1})$ $= \lambda_\mu^+ (A^{-1}XY^{-1}A)$, by the definition
$= \lambda_\mu^+ (A^{-1}XAA^{-1}Y^{-1}A)$
$= \lambda_\mu^+ ((A^{-1}XA)(A^{-1}Y^{-1}A))$
$= \lambda_\mu^+ ((A^{-1}XA)(A^{-1}Y^{-1}A)^{-1})$
$\leq \max \{ \lambda_\mu^+ (A^{-1}XA), \lambda_\mu^+ (A^{-1}Y^{-1}A)^{-1}\}$
$\leq \max \{ \lambda_\mu^+ (A^{-1}XA), \lambda_\mu^+ (A^{-1}Y^{-1}A)\}$,
(since λ_μ is a bipolar anti fuzzy subgroup of ϑ)
$\leq \max \{(A\lambda_\mu^+ A^{-1})(X), (A\lambda_\mu^+ A^{-1})(Y)\}$.

Therefore, $(A\lambda_\mu^+ A^{-1})(XY^{-1}) \leq \max \{(A\lambda_\mu^+ A^{-1})(X), (A\lambda_\mu^+ A^{-1})(Y)\}$.

ii. $(A\lambda_\mu^- A^{-1})(XY^{-1})$ $= \lambda_\mu^- (A^{-1}XY^{-1}A)$, by the definition
$= \lambda_\mu^- (A^{-1}XAA^{-1}Y^{-1}A)$
$= \lambda_\mu^- ((A^{-1}XA)(A^{-1}Y^{-1}A))$
$= \lambda_\mu^- ((A^{-1}XA)(A^{-1}Y^{-1}A)^{-1})$
$\geq \min \{ \lambda_\mu^- (A^{-1}XA), \lambda_\mu^- (A^{-1}Y^{-1}A)^{-1}\}$
$\geq \min \{ \lambda_\mu^- (A^{-1}XA), \lambda_\mu^- (A^{-1}Y^{-1}A)\}$,
(since λ_μ is a bipolar anti fuzzy HX subgroup of ϑ)
$\geq \min \{(A\lambda_\mu^- A^{-1})(X), (A\lambda_\mu^- A^{-1})(Y)\}$.

Therefore, $(A\lambda_\mu^- A^{-1})(XY^{-1}) \geq \min \{(A\lambda_\mu^- A^{-1})(X), (A\lambda_\mu^- A^{-1})(Y)\}$.

Hence, $(A\lambda_\mu A^{-1}) = (A\lambda_\mu^+ A^{-1}, A\lambda_\mu^- A^{-1})$ is a bipolar anti fuzzy HX subgroup of ϑ.

2.19 Definition

Let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are two bipolar fuzzy subsets of a group G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$, $\lambda_\varphi = (\lambda_\varphi^+, \lambda_\varphi^-)$ are two bipolar fuzzy HX subgroups (bipolar anti fuzzy HX subgroups) of ϑ. Then λ_μ and λ_φ are said to be conjugate bipolar fuzzy HX subgroups (conjugate bipolar anti fuzzy HX subgroups) of ϑ if for some $A \in \vartheta$,

i. $\lambda_\mu^+(X) = \lambda_\varphi^+(A^{-1}XA)$
ii. $\lambda_\mu^-(X) = \lambda_\varphi^-(A^{-1}XA)$, for every $X \in \vartheta$.

2.20 Theorem

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\lambda_\varphi = (\lambda_\varphi^+, \lambda_\varphi^-)$ are two bipolar fuzzy HX subgroups of an abelian HX group ϑ if and only if $\lambda_\mu = \lambda_\varphi$.

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\lambda_\varphi = (\lambda_\varphi^+, \lambda_\varphi^-)$ be any two conjugate bipolar fuzzy HX subgroups of an abelian HX group ϑ, then for some $B \in \vartheta$,

i. $\lambda_\mu^+(A) = \lambda_\varphi^+(B^{-1}AB)$, for every $A \in \vartheta$
$\Leftrightarrow \lambda_\mu^+(A) = \lambda_\varphi^+(B^{-1}BA)$, since ϑ is abelian
$\Leftrightarrow \lambda_\mu^+(A) = \lambda_\varphi^+(EA)$
$\Leftrightarrow \lambda_\mu^+(A) = \lambda_\varphi^+(A)$

ii. $\lambda_\mu^-(A) = \lambda_\varphi^-(B^{-1}AB)$, for every $A \in \vartheta$
$\Leftrightarrow \lambda_\mu^-(A) = \lambda_\varphi^-(B^{-1}BA)$, since ϑ is abelian
$\Leftrightarrow \lambda_\mu^-(A) = \lambda_\varphi^-(EA)$
$\Leftrightarrow \lambda_\mu^-(A) = \lambda_\varphi^-(A)$

Hence, $\lambda_\mu = \lambda_\varphi \Leftrightarrow \lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\lambda_\varphi = (\lambda_\varphi^+, \lambda_\varphi^-)$ are conjugate bipolar fuzzy HX subgroups of an abelian group ϑ.

2.21 Theorem

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\lambda_\varphi = (\lambda_\varphi^+, \lambda_\varphi^-)$ are two bipolar anti fuzzy HX subgroups of an abelian HX group ϑ if and only if $\lambda_\mu = \lambda_\varphi$.

Proof : Similar to Theorem 2.20.

III. PROPERTIES OF LEVEL SUBSETS AND LOWER LEVEL SUBSETS OF A BIPOLAR FUZZY HX SUBGROUP AND BIPOLAR ANTI FUZZY HX SUBGROUP

In this section, we introduce the concept of level subsets of a bipolar fuzzy cosets of bipolar fuzzy HX subgroup and lower level subsets of a bipolar fuzzy cosets of a bipolar anti fuzzy HX subgroup and discuss some of its properties.

3.1 Definition

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of G. For any $<\alpha, \beta> \in [0,1]\times[-1,0]$, the bipolar set $U[\mu_{<\alpha,\beta>}] = \{ x \in G \,/\, \mu^+(x) \geq \alpha$ and $\mu^-(x) \leq \beta \}$ is called a level subset of the bipolar fuzzy subset μ.

3.2 Definition [7]

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of a HX group ϑ.For any $<\alpha, \beta> \in [0,1]\times[-1,0]$, we define the set $U[\lambda_{\mu<\alpha,\beta>}] = \{ A \in \vartheta \,/\, \lambda_\mu^+(A) \geq \alpha$ and $\lambda_\mu^-(A) \leq \beta \}$ is called the $<\alpha, \beta>$ level subset of λ_μ or simply the level subset of λ_μ.

3.3 Definition

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of a HX group ϑ.For any $<\alpha, \beta> \in [0,1]\times[-1,0]$, we define the set $U[A\lambda_{\mu<\alpha,\beta>}] = \{ X \in \vartheta \,/\, (A\lambda_\mu^+)(X) = \lambda_\mu^+(A^{-1}X) \geq \alpha$ and $(A\lambda_\mu^-)(X) = \lambda_\mu^-(A^{-1}X) \leq \beta$, for some $A \in \vartheta\}$ is called the level subset of a bipolar fuzzy coset $(A\lambda_\mu)$.

3.4 Definition

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of G. For any $<\alpha, \beta> \in [0,1]\times[-1,0]$, the bipolar set $L[\mu_{<\alpha,\beta>}] = \{ x \in G \,/\, \mu^+(x) \leq \alpha$ and $\mu^-(x) \geq \beta \}$ is called a lower level subset of the bipolar fuzzy subset μ.

3.5 Definition [6]

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of a HX group ϑ. For any $<\alpha, \beta> \in [0,1]\times[-1,0]$, we define the set $L[\lambda_{\mu<\alpha,\beta>}] = \{ A \in \vartheta \,/\, \lambda_\mu^+(A) \leq \alpha$ and $\lambda_\mu^-(A) \geq \beta \}$ is called the $<\alpha, \beta>$ lower level subset of λ_μ or simply the lower level subset of λ_μ.

3.6 Definition

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of a HX group ϑ. For any $<\alpha, \beta> \in [0,1]\times[-1,0]$, we define the set $L[A\lambda_{\mu<\alpha,\beta>}] = \{ X\in\vartheta \,/\, (A\lambda_\mu^+)(X) = \lambda_\mu^+(A^{-1}X) \le \alpha$ and $(A\lambda_\mu^-)(X) = \lambda_\mu^-(A^{-1}X) \ge \beta$, for some $A\in\vartheta\}$ is called the lower level subset of a bipolar fuzzy coset $(A\lambda_\mu)$.

3.7 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of ϑ then $U[A\lambda_{\mu<\alpha,\beta>}] = AU[\lambda_{\mu<\alpha,\beta>}]$ for every $A\in\vartheta$ and $<\alpha, \beta> \in [0,1]\times[-1,0]$.

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of ϑ and Let $X\in\vartheta$

Now $U[A\lambda_{\mu<\alpha,\beta>}] = U[(A\lambda_\mu^+;\alpha), (A\lambda_\mu^-;\beta)]$

Let $X\in U[A\lambda_{\mu<\alpha,\beta>}] \Rightarrow X\in U(A\lambda_\mu^+;\alpha)$ and $X\in U(A\lambda_\mu^-;\beta)$

i. $X\in U(A\lambda_\mu^+;\alpha) \Leftrightarrow (A\lambda_\mu^+)(X) \ge \alpha$

$\Leftrightarrow \lambda_\mu^+(A^{-1}X) \ge \alpha$

$\Leftrightarrow A^{-1}X \in U(\lambda_\mu^+;\alpha)$

$\Leftrightarrow X \in A\,U(\lambda_\mu^+;\alpha)$

Therefore, $U(A\lambda_\mu^+;\alpha) = A\,U(\lambda_\mu^+;\alpha)$, for every $A\in\vartheta$

ii. $X\in U(A\lambda_\mu^-;\beta) \Leftrightarrow (A\lambda_\mu^-)(X) \le \beta$

$\Leftrightarrow \lambda_\mu^-(A^{-1}X) \le \beta$

$\Leftrightarrow A^{-1}X \in U(\lambda_\mu^-;\beta)$

$\Leftrightarrow X \in A\,U(\lambda_\mu^-;\beta)$

Therefore, $U(A\lambda_\mu^-;\beta) = A\,U(\lambda_\mu^-;\beta)$, for every $A\in\vartheta$.

From these , $U[A\lambda_{\mu<\alpha,\beta>}] = A\,U[\lambda_{\mu<\alpha,\beta>}]$

3.8 Theorem

Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset on G and $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of ϑ then $L[A\lambda_{\mu<\alpha,\beta>}] = AL[\lambda_{\mu<\alpha,\beta>}]$ for every $A\in\vartheta$ and $<\alpha, \beta> \in [0,1]\times[-1,0]$.

Proof : Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy HX subgroup of ϑ and Let $X\in\vartheta$

Now $L[A\lambda_{\mu<\alpha,\beta>}] = L[(A\lambda_\mu^+;\alpha), (A\lambda_\mu^-;\beta)]$

Let $X\in L[A\lambda_{\mu<\alpha,\beta>}] \Rightarrow X\in L(A\lambda_\mu^+;\alpha)$ and $X\in L(A\lambda_\mu^-;\beta)$

i. $X\in L(A\lambda_\mu^+;\alpha) \Leftrightarrow (A\lambda_\mu^+)(X) \le \alpha$

$\Leftrightarrow \lambda_\mu^+(A^{-1}X) \le \alpha$

$\Leftrightarrow A^{-1}X \in L(\lambda_\mu^+;\alpha)$

$\Leftrightarrow X \in A\,L(\lambda_\mu^+;\alpha)$

Therefore, $L(A\lambda_\mu^+;\alpha) = A\,L(\lambda_\mu^+;\alpha)$, for every $A\in\vartheta$

ii. $X\in L(A\lambda_\mu^-;\beta) \Leftrightarrow (A\lambda_\mu^-)(X) \ge \beta$

$\Leftrightarrow \lambda_\mu^-(A^{-1}X) \ge \beta$

$\Leftrightarrow A^{-1}X \in L(\lambda_\mu^-;\beta)$

$\Leftrightarrow X \in AL(\lambda_\mu^-;\beta)$

Therefore, $L(A\lambda_\mu^-;\beta) = AL(\lambda_\mu^-;\beta)$, for every $A\in\vartheta$.

From these , $L[A\lambda_{\mu<\alpha,\beta>}] = A\,L[\lambda_{\mu<\alpha,\beta>}]$

IV. CONCLUSION

In this paper , we introduced the notion of bipolar fuzzy cosets of a bipolar fuzzy and bipolar anti fuzzy HX subgroups of a HX group with suitable examples . We extended these ideas to the bipolar fuzzy quotient HX subgroup and discussed few important properties of them.

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